数学編

Preparing terminology

- Fibonacci numbers and their companions
- Pell numbers and their companions
- Binary quintic polynomials of them
- Binary quadratic form
- Topographic form

My setting problem

- Quadratic form of Fibonacci numbers
- Topographic form of Fibonacci numbers
- Topographic form of Pell numbers
- Topographic form of Markov numbers
- Quadratic form of Markov numbers
- Quadratic form of expanded Markov numbers
- Topographic form of expanded Markov numbers

Diophantine equation

- $x^3+y^3+z^3=A$
- For given A,

  integer (x,y,z)?

- $1^3+6^3+8^3=9^3$
- $3^3+4^3+5^3=6^3$
- $9^3+(-8)^3+(-6)^3=1$
- $7^3+(-5)^3+(-6)^3=2$

Diophantine equation

- $x^3+y^3+z^3=A$
- For given A, integer (x,y,z)?

- $x^3+y^3+z^3\equiv 0,1,2,3,6,7,8 \pmod 9$

- $x^3+y^3+z^3=1 \rightarrow$ infinite
- $x^3+y^3+z^3=2 \rightarrow$ infinite
- $x^3+y^3+z^3=3 \rightarrow$ exist
- $x^3+y^3+z^3=4 \rightarrow$ not exist
- $x^3+y^3+z^3=5 \rightarrow$ not exist

- $X\equiv 0 \pmod 9 \rightarrow X^3\equiv 0$
- $X\equiv 1 \pmod 9 \rightarrow X^3\equiv 1$
- $X\equiv 2 \pmod 9 \rightarrow X^3\equiv -1$
- $X\equiv 3 \pmod 9 \rightarrow X^3\equiv 0$
- $X\equiv 4 \pmod 9 \rightarrow X^3\equiv 1$
- $X\equiv 5 \pmod 9 \rightarrow X^3\equiv -1$
- $X\equiv 6 \pmod 9 \rightarrow X^3\equiv 0$
- $X\equiv 7 \pmod 9 \rightarrow X^3\equiv 1$
- $X\equiv 8 \pmod 9 \rightarrow X^3\equiv -1$

Mordell

- $x^3+y^3+z^3=A$
- For given A, integer (x,y,z)?
- $x^3+y^3+z^3=3 \rightarrow (x, y, z)=?$

- $(x, y, z)=(1,1,1) \rightarrow 1+1+1=3$
- $(x, y, z)=(4,4,-5) \rightarrow 64+64-125=3$
- What is the third (x, y, z) ?
- Where is the deterministic algorithm ?

Mordell

- $x^3+y^3+z^3=A$
- For given A, integer (x,y,z)?
- $x^3+y^3+z^3=3 \rightarrow (x, y, z)=(1,1,1),(4,4,-5)$

- x= 569936821221962380720
- y=-472715493453327032
- z=-569936821113563493509

Matijasevic

- He solved Hilbert 10th problem negatively by using generalized Fibonacci numbers (Pell equation).
- Main product:
- Nowhere deterministic algorithm for Diophantine equation.

- Byproduct of his method:
- Matijasevic's polynomial

Matijasevic

- prime representing polynomial (1971)
- (Jones, 1976)
- $\{\pi\}=\{P \mid P>0$ for non-negative integer $a,b,c,\dots,z\}$
- $P=(k+2)\{1-[wz+h+j-q]^2-[(gk+2g+k+1)(h+j)+h-z]^2-[2n+p+q+z-e]^2-[16(k+1)^3(k+2)(n+1)^2+1-f^2]^2-[e^3(e+2)(a+1)^2+1-o^2]^2-[(a^2-1)y^2+1-x^2]^2-[16r^2y^4(a^2-1)+1-u^2]^2-[((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2]^2-[n+l+v-y]^2-[(a^2-1)l^2+1-m^2]^2-[ai+k+1-l-i]^2-[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2-[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2-[z-pl(a-p)+t(2ap-p^2-1)-pm]^2\}$

Matijasevic

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Factorial function (Wilson's theorem)

- $(p-1)! \equiv -1 \pmod{p}$, characterizing primality

- $x=\{(p-1)!+1\}/p$
- $y=p-1$
- $a=x(y+1)-(y!+1)=0$
- $\{|a^2-1|-(a^2-1)\}/2=1$
- $(y-1)\{|a^2-1|-(a^2-1)\}/2+2=p$
- $x^2-(a^2-1)y^2=1$, $x \equiv p^n+y(a-p) \pmod{2ap-p^2-1}$

Matijasevic

- prime representing polynomial (1971)
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- $\{\pi\}=\{P \mid P>0$ for non-negative integer $a,b,c,\ldots,z\}$
- $P=(k+2)\{1-[wz+h+j-q]^2-[(gk+2g+k+1)(h+j)+h-z]^2-[2n+p+q+z-e]^2-[16(k+1)^3(k+2)(n+1)^2+1-f^2]^2-[e^3(e+2)(a+1)^2+1-o^2]^2-[(a^2-1)y^2+1-x^2]^2-[16r^2y^4(a^2-1)+1-u^2]^2-[((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2]^2-[n+l+v-y]^2-[(a^2-1)l^2+1-m^2]^2-[ai+k+1-l-i]^2-[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2-[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2-[z-pl(a-p)+t(2ap-p^2-1)-pm]^2\}$

Congruence rule (Matijasevic's lemma)

- $(p-1)! \equiv -1 \pmod{p}$, characterizing primality

- $F_m=0 \pmod{F_n} \rightarrow m=kn$
- $F_m=F_{kn}=0 \pmod{F_n^2}$
- $\Leftrightarrow kF_n=0 \pmod{F_n^2}$
- $\Leftrightarrow k=0 \pmod{F_n}$
- $\Leftrightarrow m=0 \pmod{F_n}$
- $\Leftrightarrow m=0 \pmod{nF_n}$
- Pell equation; $x=T_n$, $y=U_{n-1}$
- $x^2-(a^2-1)y^2=1$, $x \equiv p^n+y(a-p) \pmod{2ap-p^2-1}$

Matijasevic

- prime representing polynomial (1971)
- (Jones, 1976)
- $\{\pi\}=\{P \mid P>0$ for non-negative integer $a,b,c,\ldots,z\}$
- $P=(k+2)\{1-[wz+h+j-q]^2-[(gk+2g+k+1)(h+j)+h-z]^2-[2n+p+q+z-e]^2-[16(k+1)^3(k+2)(n+1)^2+1-f^2]^2-[e^3(e+2)(a+1)^2+1-o^2]^2-[(a^2-1)y^2+1-x^2]^2-[16r^2y^4(a^2-1)+1-u^2]^2-[((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2]^2-[n+l+v-y]^2-[(a^2-1)l^2+1-m^2]^2-[ai+k+1-l-i]^2-[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2-[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2-[z-pl(a-p)+t(2ap-p^2-1)-pm]^2\}$

Pell equation: $x^2 - Dy^2 = \pm 1$

- $x_n = (\alpha^n + \beta^n)/2$, $y_n = (\alpha^n - \beta^n)/(2\sqrt{D})$, non-square D
- Generalized Fibonacci and Lucas numbers
- Hyperbola in (x,y) plane, $x_n/y_n \to \sqrt{D}$

D	α	β
2	$1+\sqrt{2}$	$1-\sqrt{2}$
3	$2+\sqrt{3}$	$2-\sqrt{3}$
5	$2+\sqrt{5}$	$2-\sqrt{5}$
6	$5+2\sqrt{6}$	$5-2\sqrt{6}$
7	$8+3\sqrt{7}$	$8-3\sqrt{7}$
8	$3+\sqrt{8}$	$3-\sqrt{8}$
10	$3+\sqrt{10}$	$3-\sqrt{10}$

Fibonacci numbers are generated by;

- $\{F_n\}=\{F \mid F>0$ for non-negative integer $a,b,c,\ldots,z\}$
- $F=w\{1-[u+a-l]^2-[v+b-l]^2-[l^2-lz-z^2-1]^2-[g^2-gh-h^2-1]^2-[l^2c-g]^2-[ld-m+2]^2-[(2h+g)e-m+3]^2-[x^2-mxy+y^2-1]^2-[l(p-1)-x+u]^2-[(2h+g)(r-1)-x+v]^2-[((2u-t)^2+(w-v)^2)((2u+1-t)^2+(w^2-wv-v^2-1)^2]^2\}$

Fibonacci numbers are generated by;

- $\{F_n\}=\{F \mid F>0$ for non-negative integer $a,b,c,\ldots,z\}$
- $F=w\{1-[u+a-l]^2-[v+b-l]^2-[l^2-lz-z^2-1]^2-[g^2-gh-h^2-1]^2-[l^2c-g]^2-[ld-m+2]^2-[(2h+g)e-m+3]^2-[x^2-mxy+y^2-1]^2-[l(p-1)-x+u]^2-[(2h+g)(r-1)-x+v]^2-[((2u-t)^2+(w-v)^2)((2u+1-t)^2+(w^2-wv-v^2-1)^2]^2\}$

- Based on the property of Fibonacci number

Simplified representing polynomial of Fibonacci numbers;

- binary quintic polynomial;
- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y$ (Jones, 1975)
- $F>0$ for $x=1,2,,,$: $y=1,2,,,$ $\rightarrow$ $F=F_n$
- $\{F_n\}=\{F \mid F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y>0\}$

Derive binary quintic polynomial;

- binary quintic polynomial;
- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y$ (Jones, 1975)
- $F>0$ for $x=1,2,,,: y=1,2,,, \rightarrow F=F_n$
- $\{F_n\}=\{F \mid F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y>0\}$

- Different from Matijasevic's original logic.
- Based on the property of Fibonacci number
- Set level for high school student/entrance exam.

Deal with linear sequence

- Recursively defined as linear sequence
- $u_{n+1}=Mu_n+Nu_{n-1}$

- Binary quintic polynomial for $\{u_n\}$?

Fibonacci number companions

- Recursively defined as linear sequence
- $u_{n+1}=u_n+u_{n-1}$

- $\alpha=(1+\sqrt{5})/2$, $\beta=(1-\sqrt{5})/2$
- $F_n=(\alpha^n-\beta^n)/(\alpha-\beta)$ (Fibonacci)
- $L_n= \alpha^n+\beta^n$ (Fibonacci-Lucas)

- Twins in number theory

Pell number companions

- Recursively defined as linear sequence
- $u_{n+1}=2u_n+u_{n-1}$

- $\gamma=1+\sqrt{2}$, $\delta=1-\sqrt{2}$
- $P_n=(\gamma^n-\delta^n)/(\gamma-\delta)$ (Pell)
- $Q_n=\gamma^n+\delta^n$ (Pell-Lucas)

- Twins in number theory

Tell you results at first

- $u_{n+1}=Mu_n+Nu_{n-1}$
- Generally, $\{u_n \mid u_{n+1}=Mu_n+u_{n-1}, N=1\}$ are generated by binary quintic polynomials.
- $M=1$, $N=1$ in $\{F_n\}$ and $\{L_n\}$
- $M=2$, $N=1$ in $\{P_n\}$ and $\{Q_n\}$
- $M=3$, $N=1$

- $\{F_n\}$, $\{L_n\}$, $\{P_n\}$ and $\{Q_n\}$ are generated by binary quintic polynomials.

Generalization of golden ratio

- $u_{n+1}=Mu_n+u_{n-1}$

- M=1: golden ratio (黄金比)
- $\phi=(1+\sqrt{5})/2=[1;1,1,1,1,1,\ldots]$

- M=2: silver ratio (白銀比)
- $1+\sqrt{2}=[2;2,2,2,2,2,\ldots]$

- M=3: bronze ratio (青銅比)
- $(3+\sqrt{13})/2=[3;3,3,3,3,3,\ldots]$

$$\sqrt{3}=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\cdots}}}}}}}$$

Derive binary quintic polynomial;

- binary quintic polynomial;
- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y$ (Jones, 1975)
- $F>0$ for $x=1,2,,,$: $y=1,2,,,$ → $F=F_n$
- $\{F_n\}=\{F \mid F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y>0\}$

- $x=4$ ($\neq F_i$), $y=7$ ($\neq F_j$) →

  $F=19208+5488-6272-16807-1792+8=-167$
- $F<0$, mismatch with Fibonacci number

Derive binary quintic polynomial;

- binary quintic polynomial;
- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y$ (Jones, 1975)
- $F>0$ for $x=1,2,,,: y=1,2,,, \rightarrow F=F_n$
- $\{F_n\}=\{F \mid F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y>0\}$

- $x=4$ ($\neq F_i$), $y=7$ ($\neq F_j$) $\rightarrow$
  $F=19208+5488-6272-16807-1792+8=-167<0$

- $x=2$ ($=F_3$), $y=2$ ($=F_3$) $\rightarrow$ $F=-28<0$ (mismatch)

Derive binary quintic polynomial;

- binary quintic polynomial;
- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y$ (Jones, 1975)
- $F>0$ for $x=1,2,,,$: $y=1,2,,,$ → $F=F_n$
- $\{F_n\}=\{F \mid F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y>0\}$

- $x=5$ ($=F_5$), $y=8$ ($=F_6$), (neighbor 2 terms) → $F=40960+12800-16000-32768-5000+16=8$ ($=F_6$)
- $F>0$ (match)→ $F=y=8$ (no information of $z=13=F_7$)
- $x=F_5$, $y=F_6$ → $x=F_6$, $y=F_7$ →→→ $\{F_n\}$
- recursively enumerable

Derive binary quintic polynomial;

- binary quintic polynomial;
- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y$ (Jones, 1975)
- $F>0$ for $x=1,2,,,$: $y=1,2,,,$ → $F=F_n$
- $\{F_n\}=\{F \mid F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y>0\}$

- $x=8$ ($=F_6$), $y=13$ ($=F_7$), (neighbor 2 terms) → $F=456976+140608-173056-371293-53248+26=13$
- $F>0$ (match)→ $F=y=13$ (no information of $z=21=F_8$)
- $x=F_5$, $y=F_6$ → $x=F_6$, $y=F_7$ →→→ $\{F_n\}$
- recursively enumerable

Derive binary quintic polynomial;

- binary quintic polynomial;
- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y$ (Jones, 1975)
- $F>0$ for $x=1,2,,,: y=1,2,,, \rightarrow F=F_n$
- $\{F_n\}=\{F \mid F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y>0\}$

- $x=5$ $(=F_5)$, $y=13$ $(=F_7)$, (not neighbor) $\rightarrow$
  $F=285610+54925-42250-371293-8125+26=-81107$
- $F<0$ (mismatch)
- x and y should be consecutive 2 terms ($x=F_n < y=F_{n+1}$)

Derive binary quintic polynomial;

- binary quintic polynomial;
- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y$ (Jones, 1975)
- $F>0$ for $x=1,2,,,$: $y=1,2,,,$ → $F=F_n$
- $\{F_n\}=\{F \mid F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y>0\}$

- $x=8$ ($=F_6$), $y=5$ ($=F_5$), (neighbor but reverse) → $F=10000+8000-25600-3125-20480+10=-31195$
- $F<0$ (mismatch)
- x and y should be consecutive 2 terms ($x=F_n < y=F_{n+1}$)

Derive binary quartic polynomial;

- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y$ (Jones, 1975)
- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y \leftarrow F=y$

- Essential part
- $2xy^3+x^2y^2-2x^3y-y^4-x^4+1=0$, (x,y) on quartic curve
- $\{F_n\}=\{x=F_n,\ y=F_{n+1} \mid 2xy^3+x^2y^2-2x^3y-y^4-x^4+1=0\}$

Derive binary quartic polynomial;

- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y$ (Jones, 1975)
- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y \leftarrow F=y$

- Essential part
- $2xy^3+x^2y^2-2x^3y-y^4-x^4+1=0$, (x,y) on quartic curve
- $\{F_n\}=\{x=F_n,\ y=F_{n+1} \mid 2xy^3+x^2y^2-2x^3y-y^4-x^4+1=0\}$
- $(x^2+xy-y^2)^2=1$

Derive binary quadratic polynomial;

- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y$ (Jones, 1975)
- $F=2xy^4+x^2y^3-2x^3y^2-y^5-x^4y+2y \leftarrow F=y$

- Essential part
- $2xy^3+x^2y^2-2x^3y-y^4-x^4+1=0$, (x,y) on quartic curve
- $\{F_n\}=\{x=F_n,\ y=F_{n+1} \mid 2xy^3+x^2y^2-2x^3y-y^4-x^4+1=0\}$
- $(x^2+xy-y^2)^2=1$
- $x^2+xy-y^2=\pm 1 \rightarrow$ Derive binary quadratic polynomial

Derive binary quadratic polynomial;

- $(x^2+xy-y^2)^2=1$, $x^2+xy-y^2=\pm 1$
- $2xy^3+x^2y^2-2x^3y-y^4-x^4+1=0$

- $x^2+xy-y^2=\pm 1 \rightarrow$ already shown in this talk;
- $f(p/q)=(p+q)/p$ (Diophantine approximation)
- $p^2-(p+q)q=p^2-pq-q^2=(-1)^n$, $p=F_{n+1}$, $q=F_n$
- is satisfied for consecutive 2 terms.

Derive binary quadratic polynomial;

- $x^2+xy-y^2=\pm 1$
- Consecutive 5 terms; $y-x$, x, y, $x+y$, $x+2y$ $(x<y)$
- $(y-x,x)\rightarrow \{x^2-x(y-x)-(y-x)^2\}=(x^2+xy-y^2)=(-1)^n$
- $(x,y)\rightarrow (y^2-yx-x^2)=(-x^2-xy+y^2)=-(-1)^n$
- $(y,x+y)\rightarrow \{(x+y)^2-(x+y)y-y^2\}=(x^2+xy-y^2)=(-1)^n$
- $\{(x+2y)^2-(x+2y)(x+y)-(x+y)^2\}=(-x^2-xy+y^2)=-(-1)^n$

- $(x^2+xy-y^2)^2=1$
- $2xy^3+x^2y^2-2x^3y-y^4-x^4+1=0$ (Q.E.D.)

Apology

- x^2-xy-y^2=$(-1)^n$, recommended for derivation
- x=F_{n+1}, y=F_n, (x,y) on hyperbola
- Such a derivation is easiest and I will recommend you.
- But, I couldn't notice it and I used another relation.

Apology

- I used
- $x^2-3xy+y^2=(-1)^n$, not easiest for derivation
- $x=F_n$, $y=F_{n+2}$ or $x=F_{n+2}$, $y=F_n$ (second next)

- Because,…
- I considered about $\{F_{2n-1}\}$ and $\{F_{2n}\}$ of Fibonacci numbers, relating to quadratic forms of them.
- It produced a happy accident (unexpected results).

Fibonacci numbers

Fibonacci numbers: F[n+1]=F[n]+F[n-1], F1=1, F2=1										
F[n]	1	1	2	3	5	8	13	21	34	55
F[2n-1]	**1**		**2**		**5**		**13**		**34**	
F[2n]		1		3		8		21		55
F[n]	89	144	233	377	610	987	1597	2584	4181	6765
F[2n-1]	**89**		**233**		**610**		**1597**		**4181**	
F[2n]		144		377		987		2584		6765

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F[2n-1]	1		2		5		13		34	
F[2n]		**1**		**3**		**8**		**21**		**55**
F[n]	89	144	233	377	610	987	1597	2584	4181	6765
F[2n-1]	89		233		610		1597		4181	
F[2n]		**144**		**377**		**987**		**2584**		**6765**

Fibonacci number companions

- $\alpha=(1+\sqrt{5})/2$, $\beta=(1-\sqrt{5})/2$
- $u_n=F_{2n-1}=(\alpha^{2n-1}-\beta^{2n-1})/(\alpha-\beta)$
- $u_n=F_{2n}=(\alpha^{2n}-\beta^{2n})/(\alpha-\beta)$

- Recursively defined as linear sequence
- $u_{n+1}=3u_n-u_{n-1}$

My derivation of quartic polynomial;

- $x^2-3xy+y^2=(-1)^n$
- $x=F_n$, $y=F_{n+2}$ or $x=F_{n+2}$, $y=F_n$ (second next)

- $F_n=(\alpha^n-\beta^n)/(\alpha-\beta)$
- $\alpha=(1+\sqrt{5})/2$, $\beta=(1-\sqrt{5})/2$, $\alpha+\beta=1$, $\alpha\beta=-1$, $\alpha^2+\beta^2=3$
- Easily confirmable: $x=F_n$, $y=F_{n+2} \rightarrow x^2-3xy+y^2=(-1)^n$
- Plentiful harvest: $x=F_n$, $y=F_{n+2} \leftarrow x^2-3xy+y^2=(-1)^n$
- $(2x-3y)^2-5y^2=4(-1)^n \rightarrow X^2-5Y^2=4(-1)^n$
- Pell equation, $L_n^2-5F_n^2=4(-1)^n$

My derivation of quartic polynomial;

- $x^2-3xy+y^2=(-1)^n$
- Consecutive 4 terms; x, y, x+y, x+2y   (x<y)
- x, chosen from $\{F_{2n-1}\}$; y, chosen from $\{F_{2n}\}$
- $(x,x+y) \rightarrow \{x^2-3x(x+y)+(x+y)^2\}=(-x^2-xy+y^2)=-1$
- $(y,x+2y) \rightarrow \{y^2-3y(x+2y)+(x+2y)^2\}=(x^2+xy-y^2)=1$

- x, chosen from $\{F_{2n}\}$; y, chosen from $\{F_{2n+1}\}$
- $(x,x+y) \rightarrow \{x^2-3x(x+y)+(x+y)^2\}=(-x^2-xy+y^2)=1$
- $(y,x+2y) \rightarrow \{y^2-3y(x+2y)+(x+2y)^2\}=(x^2+xy-y^2)=-1$

My derivation of quartic polynomial;

- $x^2 - 3xy + y^2 = (-1)^n$
- $x = F_n$, $y = F_{n+2}$ (second next)

- x, chosen from either $\{F_{2n-1}\}$ or $\{F_{2n}\}$
- $(x^2 + xy - y^2)^2 = 1$, consecutive 2 terms (x,y)
- $2xy^3 + x^2y^2 - 2x^3y - y^4 - x^4 + 1 = 0$ (Q.E.D.)
- $x = F_n$, $y = F_{n+1}$

Another derivation of quartic polynomial;

- For odd k
- $x^2-xy-y^2=(-1)^n$
- $x=F_{n+1}$, $y=F_n$ (next)

- $x^2-4xy-y^2=4(-1)^n$
- $x=F_{n+3}$, $y=F_n$ (third next)

- $x^2-L_kxy-y^2=(-1)^nF_k{}^2$
- $x=F_{n+k}$, $y=F_n$

Another derivation of quartic polynomial;

- For even k
- $x^2-3xy+y^2=(-1)^n$
- $x=F_n$, $y=F_{n+2}$ or $x=F_{n+2}$, $y=F_n$ (second next)

- $x^2-7xy+y^2=9(-1)^n$
- $x=F_n$, $y=F_{n+4}$ or $x=F_{n+4}$, $y=F_n$ (forth next)

- $x^2-L_kxy+y^2=(-1)^nF_k{}^2$
- $x=F_n$, $y=F_{n+k}$ or $x=F_{n+k}$, $y=F_n$
- $F_{2k-1}+F_{2k+1}=L_{2k}$

Another derivation of quartic polynomial;

- $x^2-7xy+y^2=9(-1)^n$
- $x=F_n$, $y=F_{n+4}$ (forth next)

- Consecutive 8 terms; x, y, x+y, x+2y, 2x+3y, 3x+5y, …
- x, chosen from either $\{F_{4n-3}\}$,$\{F_{4n-2}\}$ $\{F_{4n-1}\}$ or $\{F_{4n}\}$
- $(x^2+xy-y^2)^2=1$, consecutive 2 terms (x,y)
- $2xy^3+x^2y^2-2x^3y-y^4-x^4+1=0$
- $x=F_n$, $y=F_{n+1}$

Lucas numbers

Lucas numbers: L[n+1]=L[n]+L[n-1], L1=1, L2=3										
L[n]	**1**	**3**	**4**	**7**	**11**	**18**	**29**	**47**	**76**	**123**
L[2n-1]	1		4		11		29		76	
L[2n]		3		7		18		47		123
L[n]	**199**	**322**	**521**	**843**	**1364**	**2207**	**3571**	**5578**	**9349**	**15127**
L[2n-1]	199		521		1364		3571		9349	
L[2n]		322		843		2207		5578		15127

My derivation of quartic polynomial;

- $F_{n+1}F_{n-1}=(F_n)^2-(-1)^n$
- $L_{n+1}L_{n-1}=(L_n)^2-5(-1)^n$
- $x^2-3xy+y^2=-5(-1)^n$
- $x=L_n$, $y=L_{n+2}$ (second next)

- x, chosen from either $\{L_{2n-1}\}$ or $\{L_{2n}\}$
- $(x^2+xy-y^2)^2=25$
- $2xy^3+x^2y^2-2x^3y-y^4-x^4+25=0$
- $x=L_n$, $y=L_{n+1}$

Pell numbers

Pell numbers, P[n+1]=2P[n]+P[n-1], P1=1, P2=2										
P[n]	**1**	**2**	**5**	**12**	**29**	**70**	**169**	**408**	**985**	**2378**
P[2n-1]	1		5		29		169		985	
P[2n]		2		12		70		408		2378
P[n]	**5741**	**13860**	**33461**	**80782**	**195025**	**470832**	**1136689**	**2744210**	**6625109**	**15994428**
P[2n-1]	5741		33461		195025		1136689		6625109	
P[2n]		13860		80782		470832		2744210		15994428

Pell numbers

- $\{P_n\}=\{1,2,5,12,29,70,169,408,985,2378,,,\}$
- quasi-exponential, $\sim(1+\sqrt{2})^n$

- $f(p/q)=(p+2q)/(p+q)$ (Diophantine approximation)
- $p_{n+1}=p_n+2q_n$, $q_{n+1}=p_n+q_n$
- $1/1, 3/2, 7/5, 17/12, 41/29, 99/70, 239/169,,, \to \sqrt{2}$
- Denominator of best approximating fraction

My derivation of quartic polynomial;

- $x^2-6xy+y^2=4(-1)^n$
- $x=P_n$, $y=P_{n+2}$ (second next)

- $x^2-34xy+y^2=144(-1)^n$
- $x=P_n$, $y=P_{n+4}$ (forth next)

- $x^2-Q_{2k}xy+y^2=(-1)^n P_{2k}{}^2$
- $x=P_n$, $y=P_{n+2k}$
- $P_{2k-1}+P_{2k+1}=Q_{2k}$

My derivation of quartic polynomial;

- $x^2-6xy+y^2=4(-1)^n$
- Consecutive 4 terms; x, y, x+2y, 2x+5y
- x, chosen from $\{P_{2n-1}\}$; y, chosen from $\{P_{2n}\}$
- $\{x^2-6x(x+2y)+(x+2y)^2\}=(-4x^2-8xy+4y^2)=-4$
- $\{y^2-6y(2x+5y)+(2x+5y)^2\}=(4x^2+8xy-4y^2)=4$

- x, chosen from $\{P_{2n}\}$; y, chosen from $\{P_{2n+1}\}$
- $\{x^2-6x(x+2y)+(x+2y)^2\}=(-4x^2-8xy+4y^2)=4$
- $\{y^2-3y(x+2y)+(x+2y)^2\}=(4x^2+8xy-4y^2)=-4$

My derivation of quartic polynomial;

- $x^2-6xy+y^2=4(-1)^n$
- $x=P_n$, $y=P_{n+2}$ (second next)

- x, chosen from either $\{P_{2n-1}\}$ or $\{P_{2n}\}$
- $(x^2+2xy-y^2)^2=1$, consecutive 2 terms (x,y)
- $4xy^3-2x^2y^2-4x^3y-y^4-x^4+1=0$
- $x=P_n$, $y=P_{n+1}$

Results

- $u_{n+1}=Mu_n+Nu_{n-1}$
- Generally, $\{u_n \mid u_{n+1}=Mu_n+u_{n-1}, N=1\}$ are generated by binary quintic polynomials.
- $M=1, N=1$ in $\{F_n\}$ and $\{L_n\}$
- $M=2, N=1$ in $\{P_n\}$ and $\{Q_n\}$

- $\{F_n\}$, $\{L_n\}$, $\{P_n\}$ and $\{Q_n\}$ are generated by binary quintic polynomials.

Results

- $u_{n+1}=Mu_n+u_{n-1}$, $u_1=L$, $u_2=N$, $u_3=MN+L$
- M=1, L=1, N=1 (Fibonacci)
- M=1, L=1, N=3 (Lucas)
- M=2, L=1, N=2 (Pell)
- M=2, L=2, N=6 (Pell-Lucas)

- $x^2-(M^2+2)xy+y^2=\{M^2(L^2-N^2)+M^3LN\}(-1)^n$
- $x=u_n$, $y=u_{n+2}$ (second next)

- Quartic polynomial: $(x^2+Mxy-y^2)^2=(L^2-N^2+MLN)^2$
- Quintic polynomial: $y(x^2+Mxy-y^2)^2=y(L^2-N^2+MLN)^2$
- $x=u_n$, $y=u_{n+1}$

Results, unexpected but extended

- $u_{n+1}=Mu_n+u_{n-1}$, $u_1=L$, $u_2=N$, $u_3=MN+L$
- $M=1$, $L=1$, $N=1$ (Fibonacci)
- $M=1$, $L=1$, $N=3$ (Lucas)
- $M=2$, $L=1$, $N=2$ (Pell)
- $M=2$, $L=2$, $N=6$ (Pell-Lucas)

- $x^2-(M^2+2)xy+y^2=\{M^2(L^2-N^2)+M^3LN\}(-1)^n$
- $x=u_n$, $y=u_{n+2}$ (second next)

- Quartic polynomial: $(x^2+Mxy-y^2)^2=(L^2-N^2+MLN)^2$
- Quintic polynomial: $y(x^2+Mxy-y^2)^2=y(L^2-N^2+MLN)^2$
- $x=u_n$, $y=u_{n+1}$

Plentiful harvest of Diophantine equation

- $(x^2+y^2+1)/xy=3 \leftarrow\rightarrow x=F_{2n-1},\ y=F_{2n+1}$
- $(x^2+y^2-1)/xy=3 \leftarrow\rightarrow x=F_{2n},\ y=F_{2n+2}$
- $(x^2+y^2-5)/xy=3 \leftarrow\rightarrow x=L_{2n-1},\ y=L_{2n+1}$
- $(x^2+y^2+5)/xy=3 \leftarrow\rightarrow x=L_{2n},\ y=L_{2n+2}$
- $(x^2+y^2+4)/xy=6 \leftarrow\rightarrow x=P_{2n-1},\ y=P_{2n+1}$
- $(x^2+y^2-4)/xy=6 \leftarrow\rightarrow x=P_{2n},\ y=P_{2n+2}$
- $(x^2+y^2-32)/xy=6 \leftarrow\rightarrow x=Q_{2n-1},\ y=Q_{2n+1}$
- $(x^2+y^2+32)/xy=6 \leftarrow\rightarrow x=Q_{2n},\ y=Q_{2n+2}$

- $\{x+y+xy(x+y)\}\{1/x+1/y+1/xy(x+y)\} =(x+y)^2+6 \leftarrow\rightarrow x=F_{2n-1},\ y=F_{2n+1}$
- $\{x+y+xy(x+y)\}\{1/x+1/y+4/xy(x+y)\} =(x+y)^2+12 \leftarrow\rightarrow x=P_{2n-1},\ y=P_{2n+1}$

Plentiful harvest of Diophantine equation

- $(x^2+y^2+1)/xy=3 \leftarrow\rightarrow x=F_{2n-1}, y=F_{2n+1}$
- $(x^2+y^2+4)/xy=6 \leftarrow\rightarrow x=P_{2n-1}, y=P_{2n+1}$

Plentiful harvest of Diophantine equation

- $(x^2+y^2+1)/xy=3 \leftarrow\rightarrow x=F_{2n-1}, y=F_{2n+1}$
- $(x^2+y^2+4)/xy=6 \leftarrow\rightarrow x=P_{2n-1}, y=P_{2n+1}$

- Markov equation
- $x^2+y^2+z^2=3xyz \leftarrow x=F_{2n-1}, y=F_{2n+1}, z=1$
- $x^2+y^2+z^2=3xyz \leftarrow x=P_{2n-1}, y=P_{2n+1}, z=2$

- This is the reason why Fibonacci and Pell numbers appear in Markov number. (later discussed)

Plentiful harvest of Diophantine equation

- $(x^2+y^2+1)/xy=3 \leftarrow\rightarrow x=F_{2n-1}, y=F_{2n+1}$
- $(x^2+y^2+4)/xy=6 \leftarrow\rightarrow x=P_{2n-1}, y=P_{2n+1}$

- Markov equation
- $x^2+y^2+z^2=3xyz \leftarrow x=F_{2n-1}, y=F_{2n+1}, z=1$
- $x^2+y^2+z^2=3xyz \leftarrow x=P_{2n-1}, y=P_{2n+1}, z=2$

- non$\{F_{2n-1}\}$ non$\{P_{2n-1}\}$ also appears in Markov number.
- $x^2+y^2+z^2=3xyz \leftarrow (x^2+y^2+9)/xy=9, z=3$
- $x^2+y^2+z^2=3xyz \leftarrow (x^2+y^2+16)/xy=12, z=4$

Results

- $\{F_n\}$, $\{L_n\}$, $\{P_n\}$ and $\{Q_n\}$ are representable by quintic polynomials.
- $\{F_n\}$, $\{L_n\}$, $\{P_n\}$ and $\{Q_n\}$ are representable by quadratic curves.

- Next, I will show you
- $\{F_{2n-1}\}$, $\{F_{2n}\}$, $\{L_{2n-1}\}$, $\{L_{2n}\}$ are representable by quadratic surfaces.
- $\{P_{2n-1}\}$, $\{P_{2n}\}$, $\{Q_{2n-1}\}$, $\{Q_{2n}\}$ are representable by topographs.

Preparing terminology

- Fibonacci numbers and their companions
- Pell numbers and their companions
- Binary quintic polynomials of them
- Binary quadratic form
- Topographic form

Can you hear the shape of $3x^2+6xy-5y^2$?

- Conway’s lecture note;

  “The sensual quadratic form”, Springer readings

Can you hear the shape of $3x^2+6xy-5y^2$?

- Conway's lecture note;

  "The sensual quadratic form", Springer readings

- $3x^2+6xy-5y^2=z$
- For given z, integer (x,y)?

- $3x^2+6xy-5y^2=4$, integer (x,y)=(1,1)
- $3x^2+6xy-5y^2=7$, integer (x,y) ?

Can you hear the shape of $3x^2+6xy-5y^2$?

- Conway's lecture note;
  "The sensual quadratic form", Springer readings

- $3x^2+6xy-5y^2=z$
- For given z, integer (x,y)?

- $3x^2+6xy-5y^2=4$, integer $(x,y)=(1,1)$
- $3x^2+6xy-5y^2=7$, integer (x,y) cannot exist ?

My answer for $3x^2+6xy-5y^2=7$

- $3x^2+6xy-5y^2=3(x+y)^2-8y^2$
- $X=x+y$, $Y=y$, $3x^2+6xy-5y^2=3X^2-8Y^2$
- $\{(x,y)\}=\{(X,Y)\}$

- $X\equiv 0 \pmod 8 \rightarrow X^2\equiv 0 \rightarrow 3X^2\equiv 0$
- $X\equiv 1 \pmod 8 \rightarrow X^2\equiv 1 \rightarrow 3X^2\equiv 3$
- $X\equiv 2 \pmod 8 \rightarrow X^2\equiv 4 \rightarrow 3X^2\equiv 4$
- $X\equiv 3 \pmod 8 \rightarrow X^2\equiv 1 \rightarrow 3X^2\equiv 3$
- $X\equiv 4 \pmod 8 \rightarrow X^2\equiv 0 \rightarrow 3X^2\equiv 0$
- $X\equiv 5 \pmod 8 \rightarrow X^2\equiv 1 \rightarrow 3X^2\equiv 3$
- $X\equiv 6 \pmod 8 \rightarrow X^2\equiv 0 \rightarrow 3X^2\equiv 0$
- $X\equiv 7 \pmod 8 \rightarrow X^2\equiv 1 \rightarrow 3X^2\equiv 3$

- $3x^2+6xy-5y^2=3(x+y)^2-8y^2$
- $X=x+y$, $Y=y$, $3x^2+6xy-5y^2=3X^2-8Y^2$
- $\{(x,y)\}=\{(X,Y)\}$

- $Y\equiv 0 \pmod 8 \rightarrow Y^2\equiv 0 \rightarrow 8Y^2\equiv 0$
- $Y\equiv 1 \pmod 8 \rightarrow Y^2\equiv 1 \rightarrow 8Y^2\equiv 0$
- $Y\equiv 2 \pmod 8 \rightarrow Y^2\equiv 4 \rightarrow 8Y^2\equiv 0$
- $Y\equiv 3 \pmod 8 \rightarrow Y^2\equiv 1 \rightarrow 8Y^2\equiv 0$
- $Y\equiv 4 \pmod 8 \rightarrow Y^2\equiv 0 \rightarrow 8Y^2\equiv 0$
- $Y\equiv 5 \pmod 8 \rightarrow Y^2\equiv 1 \rightarrow 8Y^2\equiv 0$
- $Y\equiv 6 \pmod 8 \rightarrow Y^2\equiv 0 \rightarrow 8Y^2\equiv 0$
- $Y\equiv 7 \pmod 8 \rightarrow Y^2\equiv 1 \rightarrow 8Y^2\equiv 0$

My answer for $3x^2+6xy-5y^2=7$

- $3x^2+6xy-5y^2=3(x+y)^2-8y^2$
- $X=x+y$, $Y=y$, $3x^2+6xy-5y^2=3X^2-8Y^2$
- $\{(x,y)\}=\{(X,Y)\}$ after diagonalization, $3X^2-8Y^2=7$

- $3X^2-8Y^2\equiv 0,3,4 \pmod 8$
- $3x^2+6xy-5y^2=7$ (inconsistent)

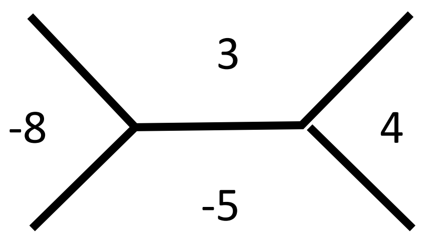
Conway's answer for $3x^2+6xy-5y^2=7$

- $3x^2+6xy-5y^2=z$
- For given z, integer (x,y)?
- $3x^2+6xy-5y^2=7$, integer (x,y)?

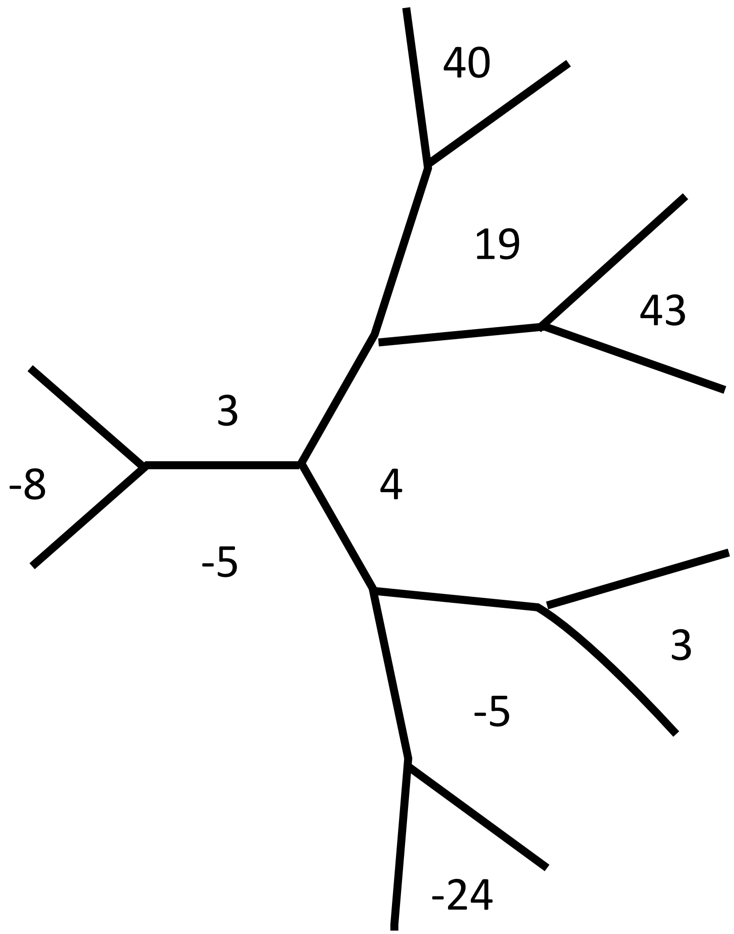
- Topograph (tree)
- $f(x,y)=3x^2+6xy-5y^2$
- $f(1,0)=3$, $f(0,1)=-5$, $f(1,1)=4$, $f(1,-1)=-8$

Topograph for $f(x,y)=3x^2+6xy-5y^2$

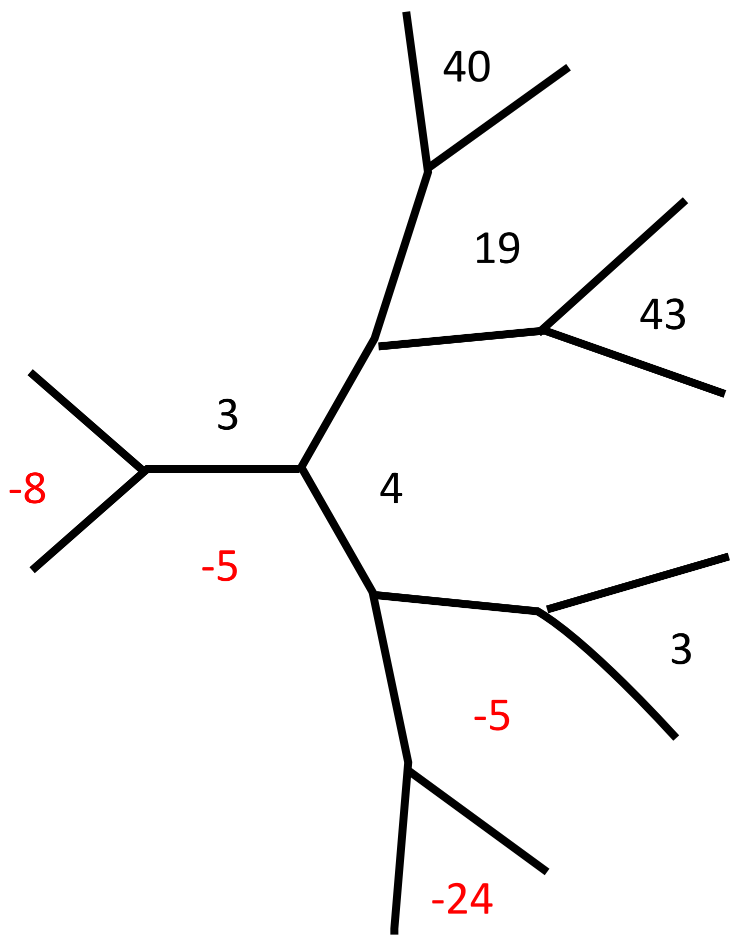
- $f(1,0)=3$, $f(0,1)=-5$, $f(1,1)=4$, $f(1,-1)=-8$



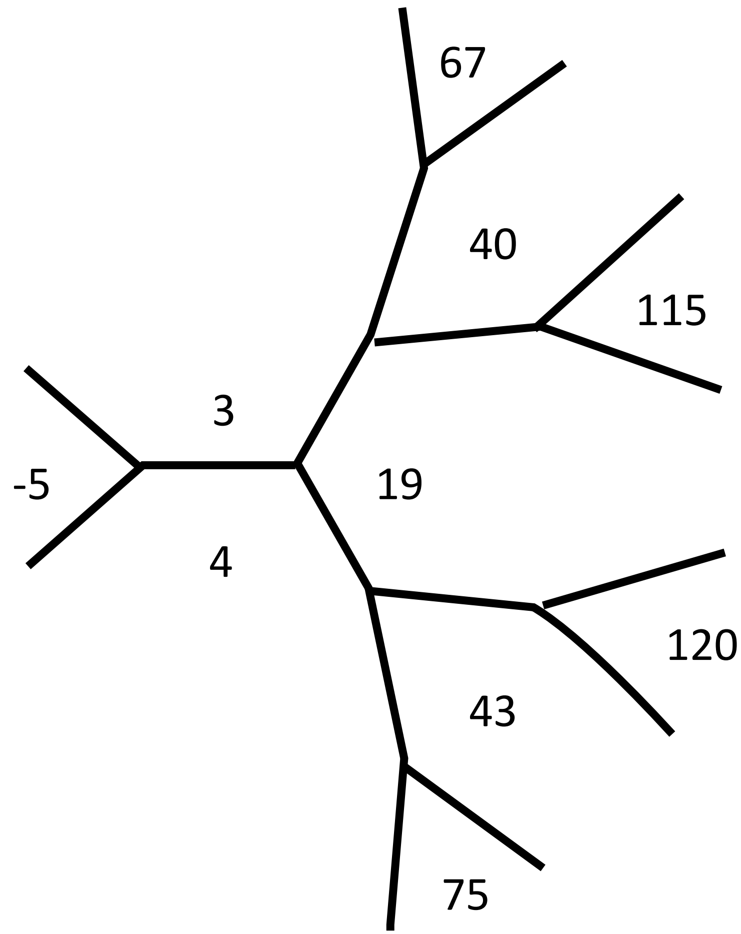
Topograph for $f(x,y)=3x^2+6xy-5y^2$



Topograph for $f(x,y)=3x^2+6xy-5y^2$



Topograph for $f(x,y)=3x^2+6xy-5y^2$



Conway's answer for $3x^2+6xy-5y^2=7$

- $3x^2+6xy-5y^2=z$
- For given z, integer (x,y)?
- $3x^2+6xy-5y^2=7$, integer (x,y)?

- Integer (x,y) cannot exist for 7.

Conway's answer for $3x^2+6xy-5y^2=7$

- Integer (x,y) exist for 4,40,100,120, (0 mod 4)
- Integer (x,y) exist for 3,19,43,67,75,115, (3 mod 4)

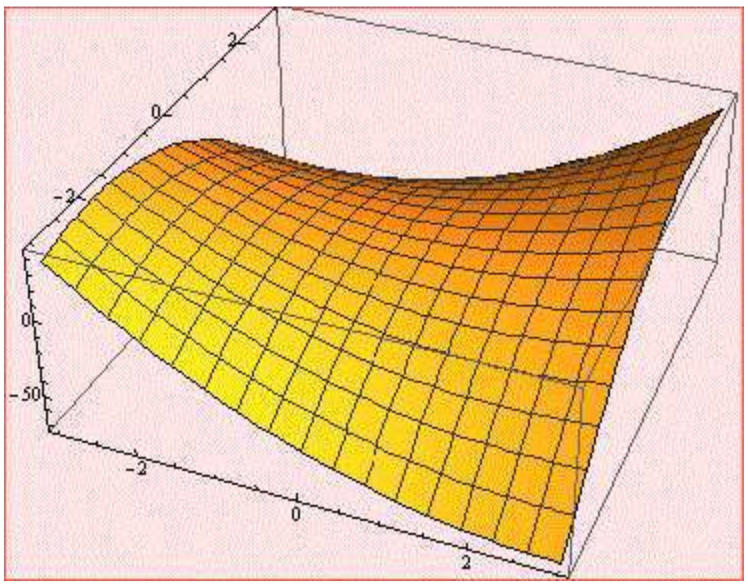
- Integer (x,y) exist for 3,75,120, (0 mod 3)
- Integer (x,y) exist for 4,19,40,43,67,100,115, (1 mod 3)

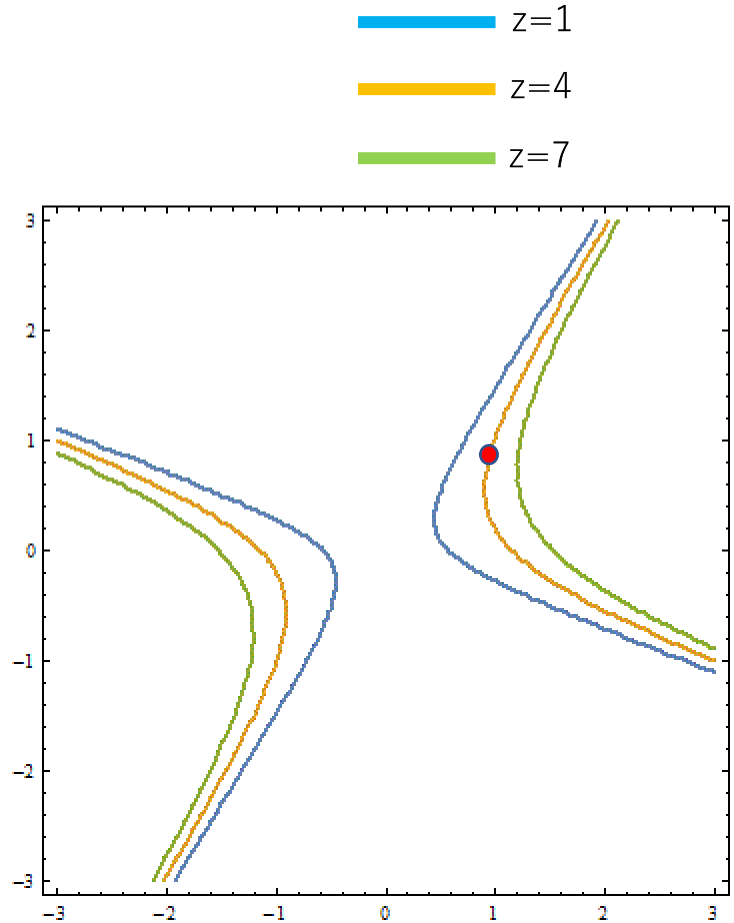
- Integer (x,y) exist for 40,120, (0 mod 8)
- Integer (x,y) exist for 4,100, (4 mod 8)
- Integer (x,y) exist for 3,19,43,67,75,115, (3 mod 8)

- Integer (x,y) cannot exist for 7.

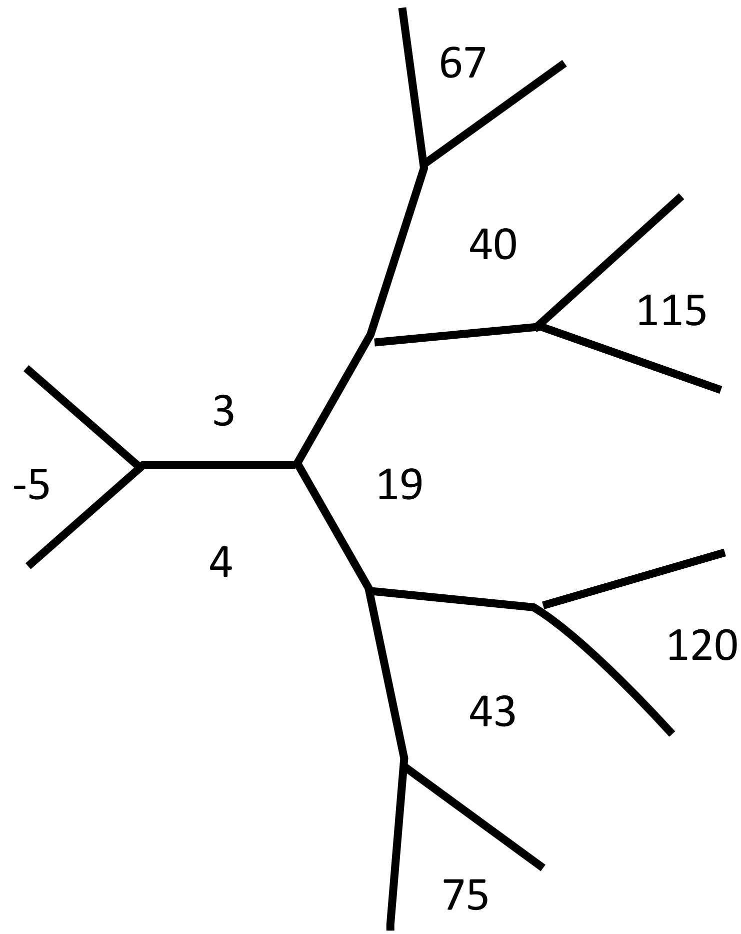
Hyperbolic paraboloid surface

- $z=3x^2+6xy-5y^2$
- For given z, integer (x,y)?

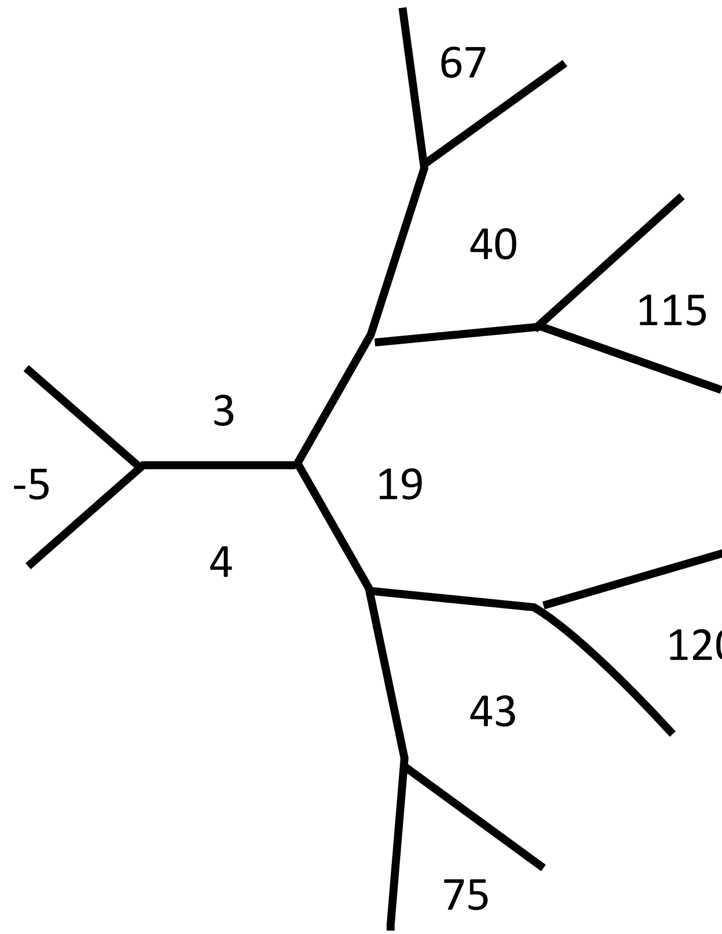




Topograph for $f(x,y)=3x^2+6xy-5y^2$



The simple manner seems to me mysterious, almost magic



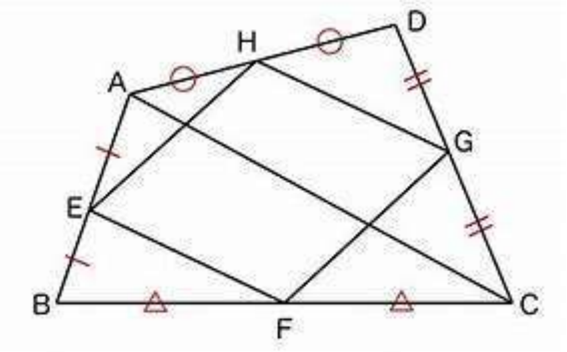
Property of binary quadratic form

- $f(x,y)=ax^2+bxy+cy^2$
- $f(x_1+x_2,y_1+y_2)+f(x_1-x_2,y_1-y_2)=2\{f(x_1,y_1)+f(x_2,y_2)\}$

- $\mathbf{v}_1=(x_1,y_1)$, $\mathbf{v}_2=(x_2,y_2)$
- $f(\mathbf{v}_1+\mathbf{v}_2)+f(\mathbf{v}_1-\mathbf{v}_2)=2\{f(\mathbf{v}_1)+f(\mathbf{v}_2)\}$
- $f(a\mathbf{v})=a^2f(\mathbf{v})$

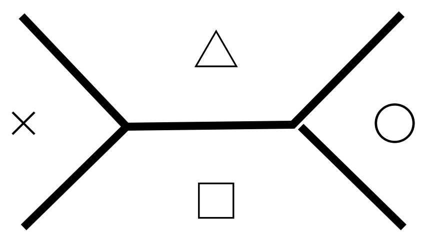
Property of binary quadratic form

- $f(x,y)=ax^2+bxy+cy^2$
- $f(x_1+x_2,y_1+y_2)+f(x_1-x_2,y_1-y_2)=2\{f(x_1,y_1)+f(x_2,y_2)\}$

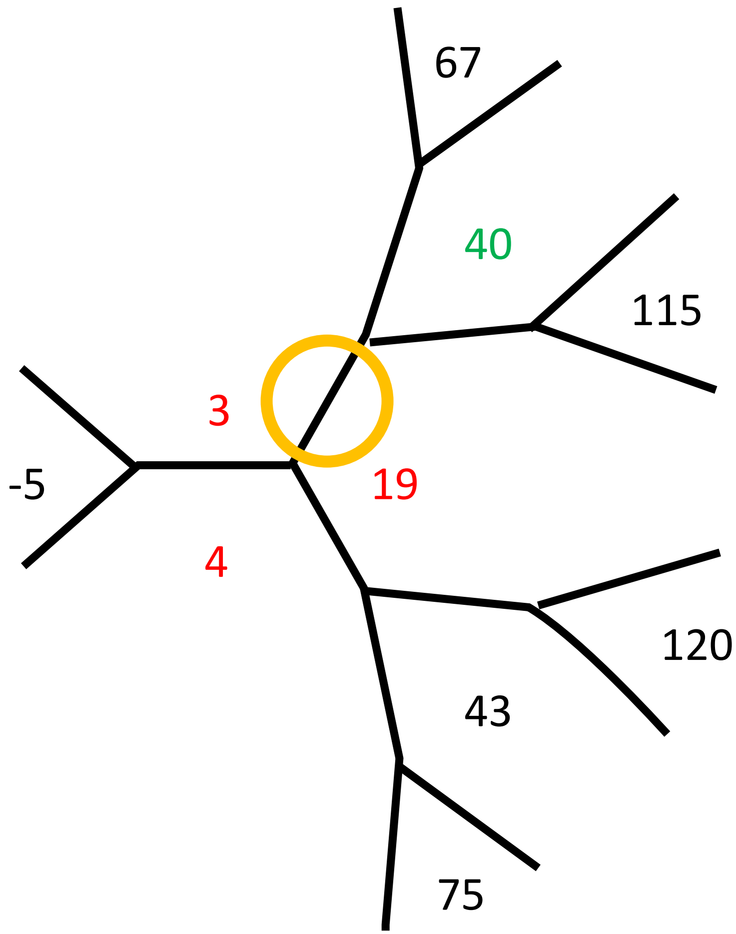


- $\mathbf{v}_1=(x_1,y_1)$, $\mathbf{v}_2=(x_2,y_2)$
- $f(\mathbf{v}_1+\mathbf{v}_2)+f(\mathbf{v}_1-\mathbf{v}_2)=2\{f(\mathbf{v}_1)+f(\mathbf{v}_2)\}$
- $f(a\mathbf{v})=a^2f(\mathbf{v})$
- Quadratic surface version of midpoint connector theorem (parallelogram)
- not satisfied for $f(x,y)=ax^2+bxy+cy^2+dx+ey+f$

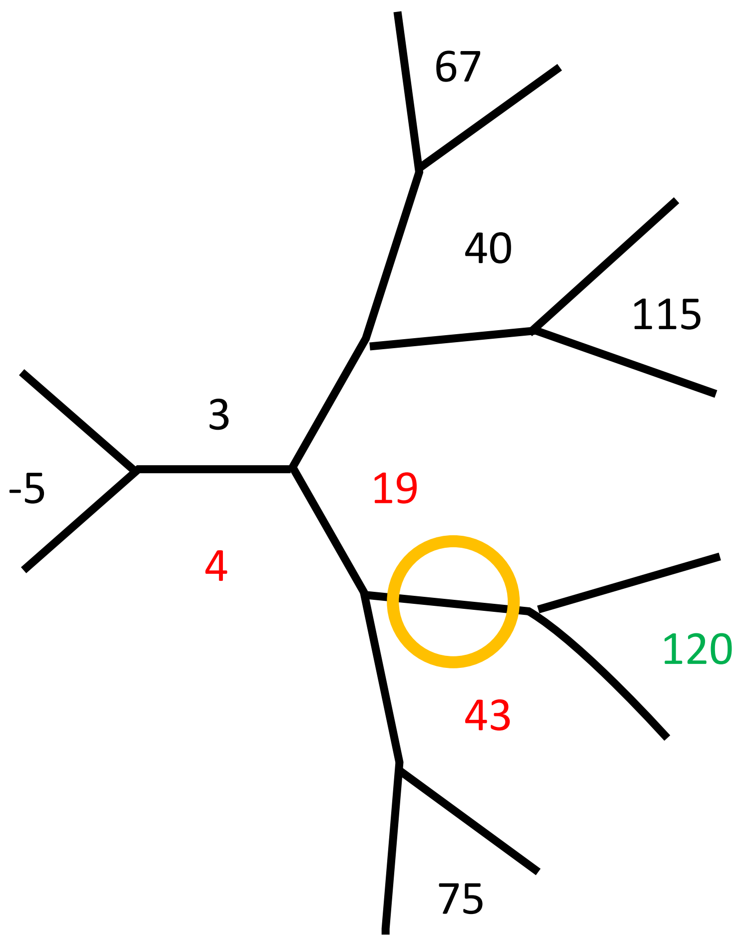
○ + × = 2(△ + □) scheme



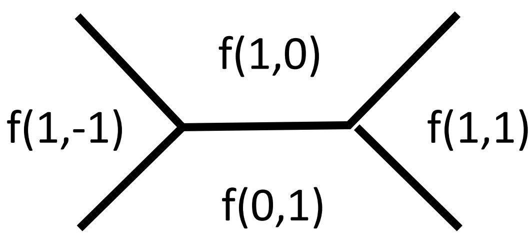
$4+40=2(3+19)$



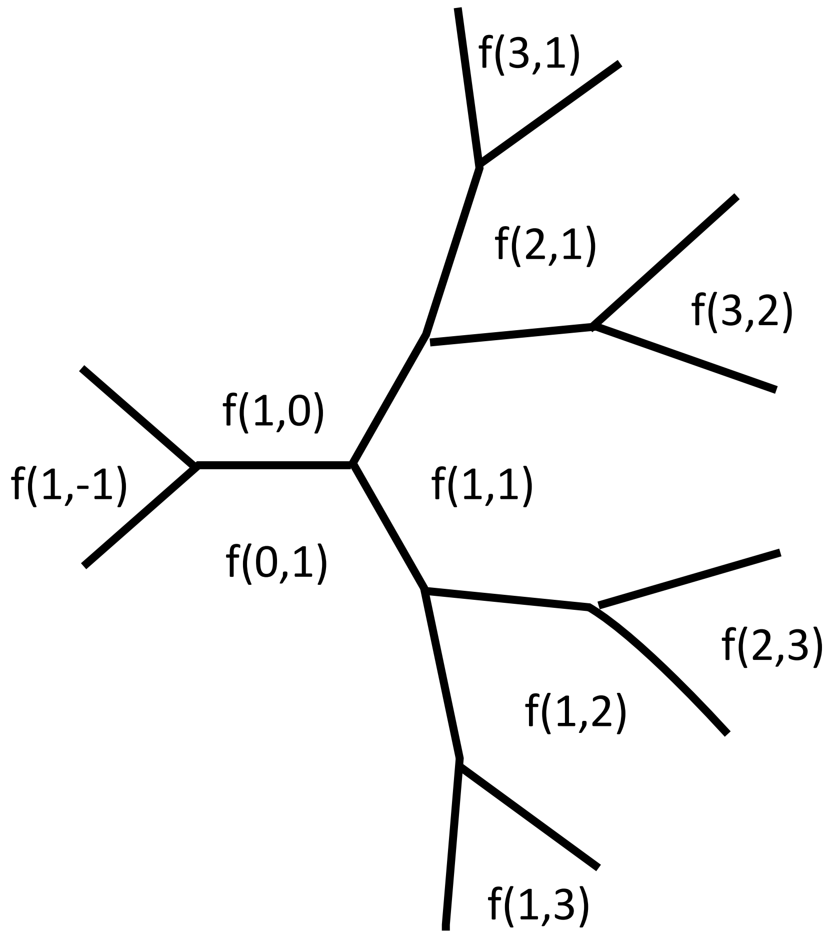
$$4+120=2(19+43)$$



Topograph for $f(x,y)=ax^2+bxy+cy^2$



Topograph for $f(x,y)=ax^2+bxy+cy^2$



Now on setting problem

- Quadratic form of Fibonacci numbers
- Topographic form of Fibonacci numbers
- Topographic form of Pell numbers
- Topographic form of Markov numbers
- Quadratic form of Markov numbers
- Quadratic form of expanded Markov numbers
- Topographic form of expanded Markov numbers

Can you hear the shape of $3x^2+6xy-5y^2$?

- Conway's lecture note;

  "The sensual quadratic form", Springer readings
- $3x^2+6xy-5y^2=4$, integer $(x,y)=(1,1)$
- $3x^2+6xy-5y^2=7$, integer (x,y) cannot exist

- $x^2-xy+y^2=1,3,4,7$, integer (x,y) exist
- $x^2-xy+y^2=2,5,6,8$, integer (x,y) cannot exist

- $2x^2+xy+4y^2=31$, integer (x,y) cannot exist

Setting problem: quadratic form

- Conway
- Binary quadratic form, $z=ax^2+bxy+cy^2$ (a priori)
- Simultaneously quadratic, x^2, xy and y^2
- not including x, y or constant
- For given z, integer (x,y)?, (a,b,c)→(x,y)

- Inverse problem
- $f(x,y)=ax^2+bxy+cy^2$ (a posteriori)
- Determine shape parameters (a,b,c) of quadratic surface where infinite sequence $\{u_n\}$ locates on lattice points. $u_n=f(x,y)\rightarrow(a,b,c)$

$\{u_n\}$, represented as quadratic form

- $f(x,y)=ax^2+bxy+cy^2$
- Simultaneously quadratic, x^2, xy and y^2
- not including x, y or constant
- as such $f(x,y)=ax^2+bxy+cy^2+dx+ey+f$

- $\{u_n\}$; $u_{n+1}=Mu_n+Nu_{n-1}$
- Input x and y, then output $f(x,y)=u_n$, $\{u_n\}<\{f(x,y)\}$

Necessary/sufficient condition for $\{u_n\}$, represented as quadratic form

- $\{u_n\}$; $u_{n+1}=Mu_n+Nu_{n-1}$
- $u_{n+1}=Mu_n+Nu_{n-1}=2(u_n+u_{n-1})-u_{n-2}$ ←midpoint connect
- $(M-2)u_n=(2-N)u_{n-1}-u_{n-2}=M(M-2)u_{n-1}+N(M-2)u_{n-2}$
- $M(M-2)=(2-N)$
- $N(M-2)=-1$

- $M=1,N=1$ → NG
- $M=3,N=-1$ → OK → $u_{n+1}=3u_n-u_{n-1}$

$\{u_n\}$, represented as quadratic form

- $\{u_n\}$; $u_{n+1}=Mu_n+Nu_{n-1}$
- $M=3$, $N=-1$ → $u_{n+1}=3u_n-u_{n-1}$
- $\{f(x,y)=ax^2+bxy+cy^2\}>\{u_{n-1},u_n,u_{n+1}, \dots\}$

- $\{u_n\}$, represented as quadratic form
- $\{F_{2n-1}\}$, $\{F_{2n}\}$, $u_{n+1}=3u_n-u_{n-1}$
- $\{L_{2n-1}\}$, $\{L_{2n}\}$, $u_{n+1}=3u_n-u_{n-1}$

$u_{n+1}=3u_n-u_{n-1}$

$\{u_n\}=\{F_{2n-1}\}$のとき

- $u_{n+1}-3u_n+u_{n-1}$
- $=F_{2n+1}-3F_{2n-1}+F_{2n-3}$
- $=F_{2n}+F_{2n-1}-3F_{2n-1}+F_{2n-3}$
- $=F_{2n}-2F_{2n-1}+F_{2n-3}$
- $=F_{2n-1}+F_{2n-2}-2F_{2n-1}+F_{2n-3}$
- $=-F_{2n-1}+F_{2n-2}+F_{2n-3}$
- $=0$

$\{u_n\}=\{F_{2n}\}$のとき

- $u_{n+1}-3u_n+u_{n-1}$
- $=F_{2n+2}-3F_{2n}+F_{2n-2}$
- $=F_{2n+1}+F_{2n}-3F_{2n}+F_{2n-2}$
- $=F_{2n+1}-2F_{2n}+F_{2n-2}$
- $=F_{2n}+F_{2n-1}-2F_{2n}+F_{2n-2}$
- $=-F_{2n}+F_{2n-1}+F_{2n-2}$
- $=0$

Fibonacci Numbers

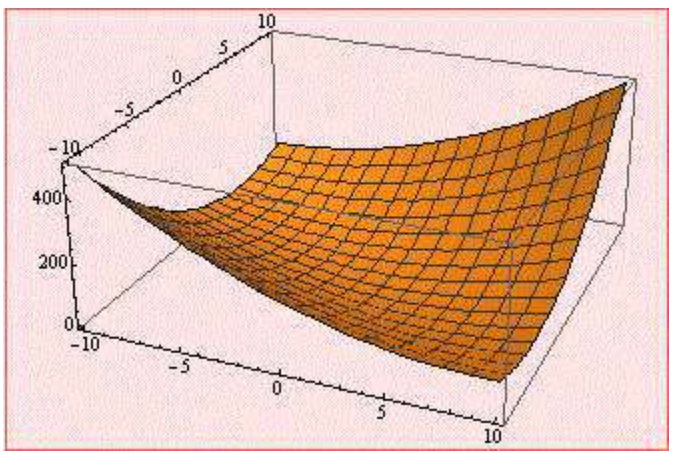
Fibonacci numbers: F[n+1]=F[n]+F[n-1], F1=1, F2=1										
F[n]	1	1	2	3	5	8	13	21	34	55
F[2n-1]	**1**		**2**		**5**		**13**		**34**	
F[2n]		1		3		8		21		55
F[n]	89	144	233	377	610	987	1597	2584	4181	6765
F[2n-1]	**89**		**233**		**610**		**1597**		**4181**	
F[2n]		144		377		987		2584		6765

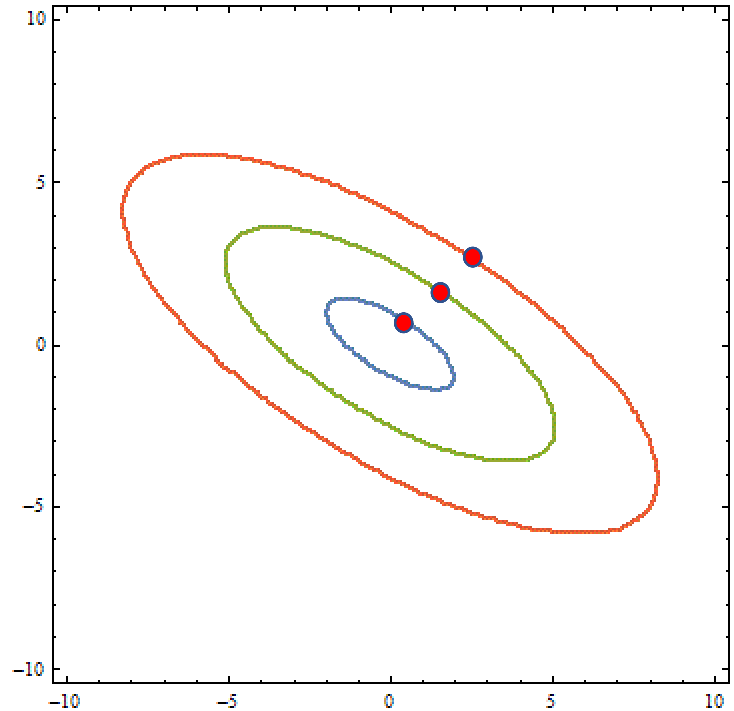
Quadratic form for $\{u_n\}=\{F_{2n-1}\}$

- $u_1=1$, $u_2=2$, $u_3=5$, $u_0=1$
- $u_1=f(1,0)=a$
- $u_2=f(0,1)=c$
- $u_3=f(1,1)=a+b+c$, $u_0=(1,-1)=a-b+c$
- $a=1$, $b=2$, $c=2$, uniquely determined
- $f(x,y)=x^2+2xy+2y^2$
- $y=x+1$, $z=5x^2+6x+2$, $z=5y^2-4y+1$, except for u_1, u_3

Elliptic paraboloid surface

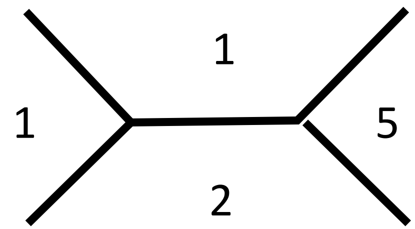
- $z=x^2+2xy+2y^2$
- $y=x+1$, $z=5x^2+6x+2$, $z=5y^2-4y+1$, except for u_1, u_3



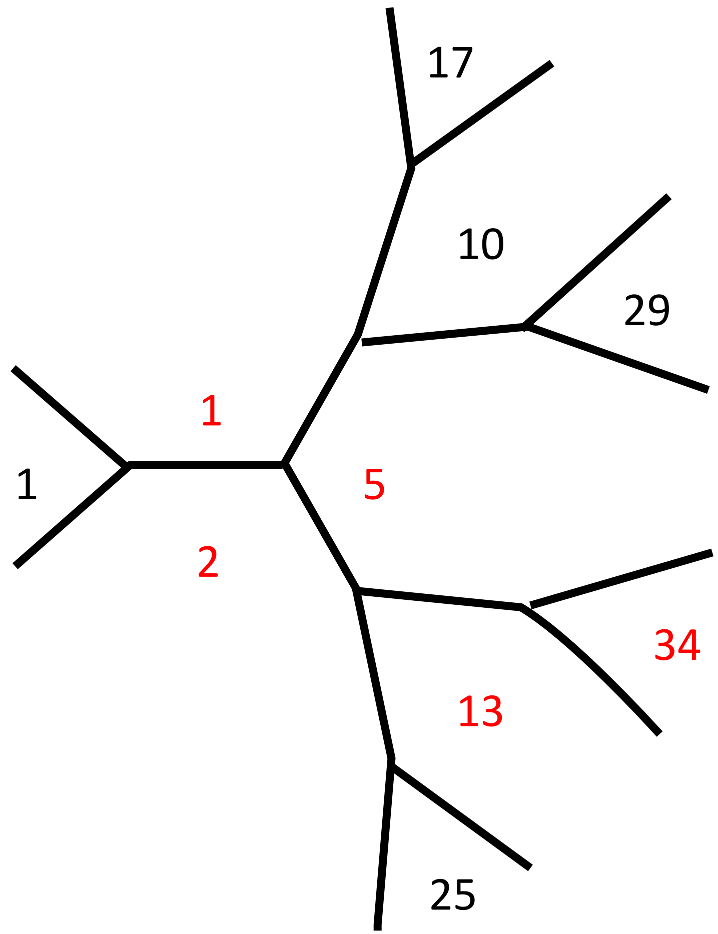


Topograph for $\{u_n\}=\{F_{2n-1}\}$

- $u_1=1$, $u_2=2$,$u_3=5$,$u_0=1$



Topograph for $\{u_n\}=\{F_{2n-1}\}$



Fibonacci Numbers

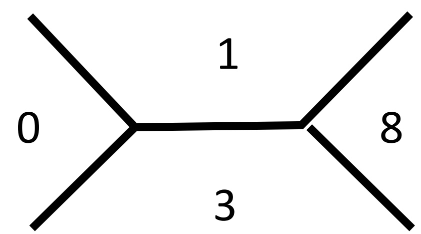
Fibonacci numbers: F[n+1]=F[n]+F[n-1], F1=1, F2=1										
F[n]	1	1	2	3	5	8	13	21	34	55
F[2n-1]	1		2		5		13		34	
F[2n]		**1**		**3**		**8**		**21**		**55**
F[n]	89	144	233	377	610	987	1597	2584	4181	6765
F[2n-1]	89		233		610		1597		4181	
F[2n]		**144**		**377**		**987**		**2584**		**6765**

Quadratic form for $\{u_n\}=\{F_{2n}\}$

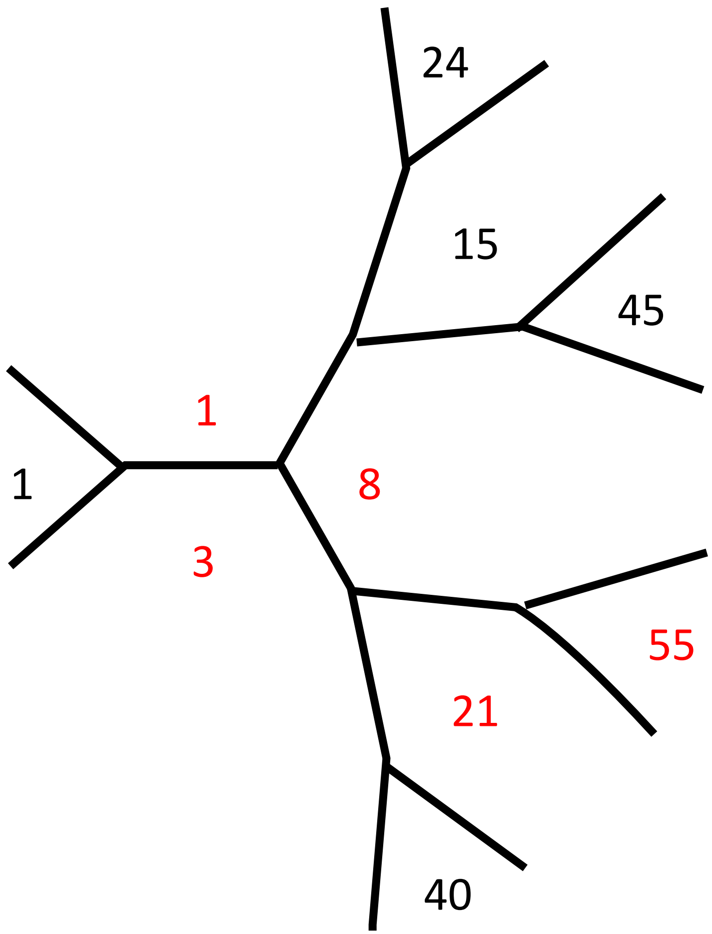
- $u_1=1$, $u_2=3$, $u_3=8$, $u_0=0$
- $u_1=f(1,0)=a$
- $u_2=f(0,1)=c$
- $u_3=f(1,1)=a+b+c$, $u_0=(1,-1)=a-b+c$
- $a=1$, $b=4$, $c=3$, uniquely determined
- $f(x,y)=x^2+4xy+3y^2$
- $y=x+1$, $z=8x^2+10x+3$, $z=8y^2-6y+1$, except for u_1, u_3

Topograph for $\{u_n\}=\{F_{2n}\}$

- $u_1=1$, $u_2=3$, $u_3=8$, $u_0=0$



Topograph for $\{u_n\}=\{F_{2n}\}$



$\{u_n\}$, not represented as quadratic form

- $\{u_n\}$; $u_{n+1}=Mu_n+Nu_{n-1}$
- $M=3$, $N=-1$ → $u_{n+1}=3u_n-u_{n-1}$
- $\{f(x,y)=ax^2+bxy+cy^2\}>\{u_{n-1},u_n,u_{n+1}, \ldots\}$

- $\{u_n\}$, not represented as quadratic form
- $\{F_n\}$, $\{L_n\}$, $u_{n+1}=u_n+u_{n-1}$
- $\{P_n\}$, $\{Q_n\}$, $u_{n+1}=2u_n+u_{n-1}$

$\{u_n\}$, not represented as quadratic form

- $\{u_n\}$; $u_{n+1}=Mu_n+Nu_{n-1}$
- $M=3$, $N=-1$ → $u_{n+1}=3u_n-u_{n-1}$
- $\{f(x,y)=ax^2+bxy+cy^2\}>\{u_{n-1},u_n,u_{n+1}, \ldots\}$

- $\{u_n\}$, not represented as quadratic form
- $\{P_{2n-1}\}$, $\{P_{2n}\}$, $\{Q_{2n-1}\}$, $\{Q_{2n}\}$, $u_{n+1}=6u_n-u_{n-1}$
- $\{F_{3n-2}\}$, $\{F_{3n-1}\}$, $\{F_{3n}\}$, $u_{n+1}=4u_n+u_{n-1}$
- $\{F_{4n-3}\}$, $\{F_{4n-2}\}$, $\{F_{4n-1}\}$, $\{F_{4n}\}$, $u_{n+1}=7u_n-u_{n-1}$

$\{u_n\}$, not represented as quadratic form

- $\{u_n\}$; $u_{n+1}=Mu_n+Nu_{n-1}$
- $M=3$, $N=-1$ → $u_{n+1}=3u_n-u_{n-1}$
- $\{f(x,y)=ax^2+bxy+cy^2\}>\{u_{n-1},u_n,u_{n+1}, \ldots\}$

- $\{u_n\}$, not represented as quadratic form
- $\{P_{2n-1}\}$, $\{P_{2n}\}$, $\{Q_{2n-1}\}$, $\{Q_{2n}\}$, $u_{n+1}=6u_n-u_{n-1}$
- $\{F_{3n-2}\}$, $\{F_{3n-1}\}$, $\{F_{3n}\}$, $u_{n+1}=4u_n+u_{n-1}$
- $\{F_{4n-3}\}$, $\{F_{4n-2}\}$, $\{F_{4n-1}\}$, $\{F_{4n}\}$, $u_{n+1}=7u_n-u_{n-1}$

Modify the scheme, $2 \to X$

- $\{u_n\}$; $u_{n+1}=Mu_n+Nu_{n-1}$
- $u_{n+1}=Mu_n+Nu_{n-1}=X(u_n+u_{n-1})-u_{n-2}$ $\leftarrow$ midpoint connect
- $(M-X)u_n=(X-N)u_{n-1}-u_{n-2}=M(M-X)u_{n-1}+N(M-X)u_{n-2}$
- $M(M-X)=(X-N)$
- $N(M-X)=-1$

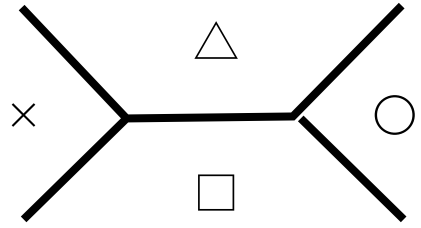
- $M=X-1, N=1 \to$ NG
- $M=X+1, N=-1 \to$ OK $\to$ $u_{n+1}=(X+1)u_n-u_{n-1}$

Modify the scheme, $2 \rightarrow X$

- $\{u_n\}$, not represented as quadratic form
- $\{F_n\}$, $\{L_n\}$, $u_{n+1}=u_n+u_{n-1}$
- $\{P_n\}$, $\{Q_n\}$, $u_{n+1}=2u_n+u_{n-1}$

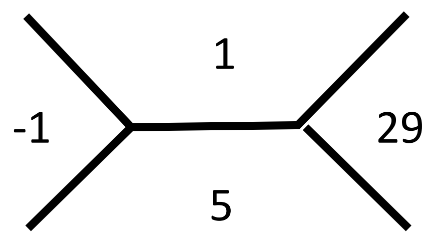
- $\{P_{2n-1}\}$, $\{P_{2n}\}$, $\{Q_{2n-1}\}$, $\{Q_{2n}\}$, $u_{n+1}=6u_n-u_{n-1}$, $X=5$
- $\{F_{3n-2}\}$, $\{F_{3n-1}\}$, $\{F_{3n}\}$, $u_{n+1}=4u_n+u_{n-1}$
- $\{F_{4n-3}\}$, $\{F_{4n-2}\}$, $\{F_{4n-1}\}$, $\{F_{4n}\}$, $u_{n+1}=7u_n-u_{n-1}$, $X=6$

○+ × =5(△+□) scheme for Pell

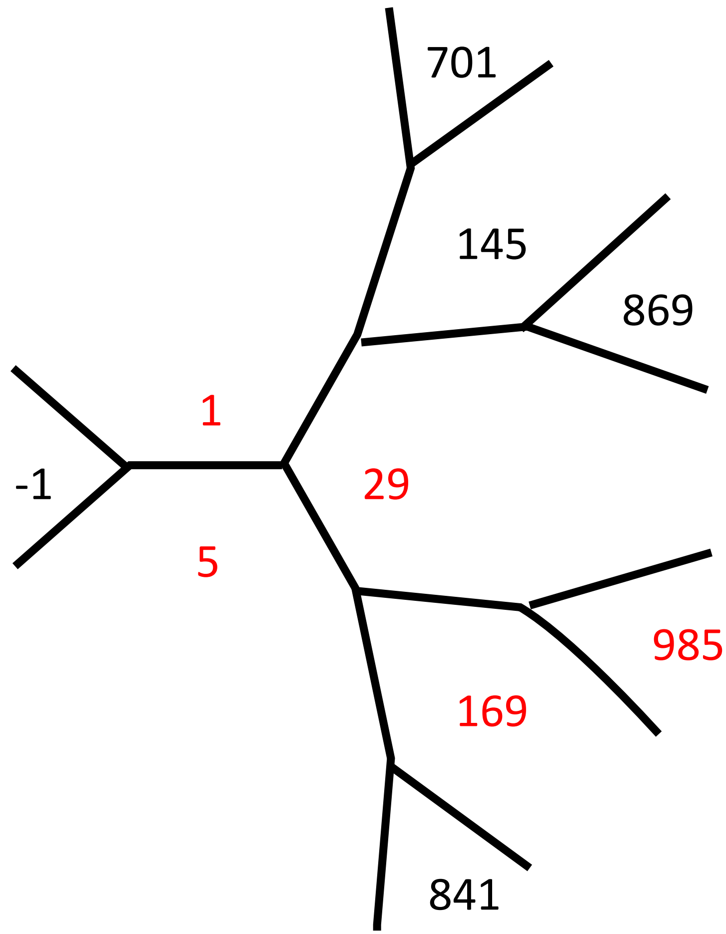


Topograph for $\{u_n\}=\{P_{2n-1}\}$

- $u_1=1$, $u_2=5$, $u_3=29$, $u_0=-1$

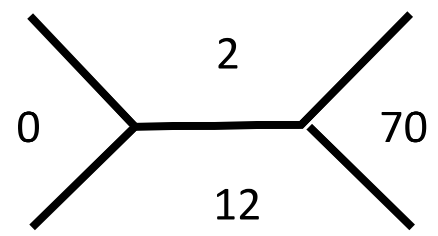


Topograph for $\{u_n\}=\{P_{2n-1}\}$

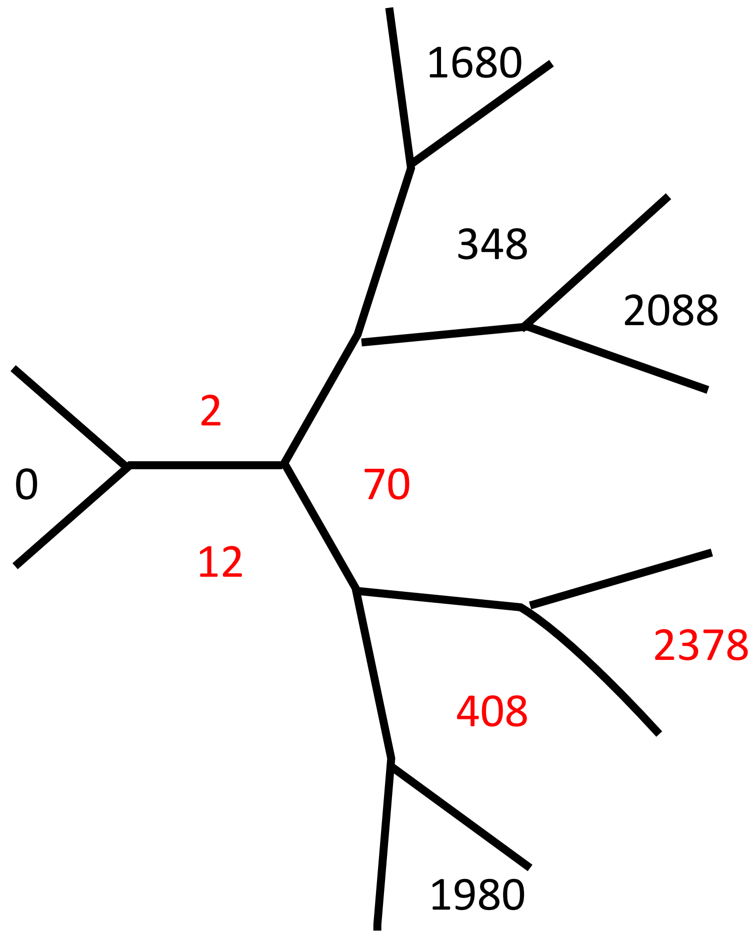


Topograph for $\{u_n\}=\{P_{2n}\}$

- $u_1=2$, $u_2=12$, $u_3=70$, $u_0=0$



Topograph for $\{u_n\}=\{P_{2n}\}$



Results

- $\{F_{2n-1}\}$, $\{F_{2n}\}$, $\{L_{2n-1}\}$, $\{L_{2n}\}$ and $\{u_n \mid u_{n+1}=3u_n-u_{n-1}\}$ are represented as binary quadratic form, $f(x,y)=ax^2+bxy+cy^2$. They are located on lattice points of elliptic paraboloid surface.

- $\{P_{2n-1}\}$, $\{P_{2n}\}$, $\{Q_{2n-1}\}$, $\{Q_{2n}\}$ and $\{u_n \mid u_{n+1}=Xu_n-u_{n-1}\}$ are not. By modifying the equality rule appropriately, they are represented as topographic form (tree).

- $\{F_n\}$, $\{L_n\}$, $\{P_n\}$, $\{Q_n\}$ and $\{u_n \mid u_{n+1}=Mu_n+u_{n-1}\}$ are generated by binary quintic polynomials.

Results

	quadratic form	topographic form
$\{F_{2n-1}\}, \{F_{2n}\}, \{L_{2n-1}\}, \{L_{2n}\}$	○	○
$\{P_{2n-1}\}, \{P_{2n}\}, \{Q_{2n-1}\}, \{Q_{2n}\}$	×	○
$\{F_n\}, \{L_n\}, \{P_n\}, \{Q_n\}$	×	×

Now on setting problem

- Quadratic form of Fibonacci numbers
- Topographic form of Fibonacci numbers
- Topographic form of Pell numbers
- Topographic form of Markov numbers
- Quadratic form of Markov numbers
- Quadratic form of expanded Markov numbers
- Topographic form of expanded Markov numbers

Fibonacci numbers

- $\{F_n\}=\{1,1,2,3,5,8,13,21,34,55,,,\}$

- $f(p/q)=(p+q)/p$ (Diophantine approximation)
- $p_{n+1}=p_n+q_n$, $q_{n+1}=p_n$
- $1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13,,, \rightarrow \phi=(1+\sqrt{5})/2$
- Denominator of best approximating fraction
- $p^2-(p+q)q=p^2-pq-q^2=(-1)^n$, $p=F_{n+1}$, $q=F_n$
- $p^2-pq-q^2\neq 0$, if $(2p-q)^2-5q^2=0$ then $\sqrt{5}$ is rational.

λ (Lagrange number) is a index of Diophantine approximation

- $|\phi - p/q| < 1/\lambda(q)^2,\ \lambda = \sqrt{5}$
- Slowest convergence, poor precision
- $|\phi - 1/1| \sim 0.618$
- $|\phi - 2/1| \sim 0.382$
- $|\phi - 3/2| \sim 0.118$
- $|\phi - 5/3| \sim 0.049$
- $|\phi - 8/5| \sim 0.018$
- $|\phi - 13/8| \sim 0.007$
- $|\phi - 21/13| \sim 0.0026$

λ (Lagrange number) is a index of Diophantine approximation

- $|\phi-p/q|<1/\lambda(q)^2$, $\lambda=\sqrt{5}$
- Slowest convergence, poor precision
- $|\phi-1/1|$~0.618
- $|\phi-2/1|$~0.382
- $|\phi-3/2|$~0.118
- $|\phi-5/3|$~0.049
- $|\phi-8/5|$~0.018
- $|\phi-13/8|$~0.007
- $|\phi-21/13|$~0.0026

- $|\pi-3/1|$ ~0.14
- $|\pi-22/7|$ ~$1/10^3$
- $|\pi-333/106|$~$8/10^5$
- $|\pi-355/113|$~$26/10^8$
- $|\pi-103993/33102|$
- $|\pi-104348/33215|$
- $|\pi-208341/66317|$
- $|\pi-312689/99532|$

$\lambda=\sqrt{5}$ is not possible to decrease any more.

- Upper bound
- $|\phi-p/q|<1/\lambda(q)^2$, $\lambda=\sqrt{5}$ holds for infinitely many p/q.
- 部分商がすべて1
- $\alpha=\phi \rightarrow \lambda=[1;1,1,1,,,]+[0;1,1,1,,,]=\phi+1/\phi=\sqrt{5}$

- Lower bound
- $|\phi-p/q|=|4q^2+4pq-4p^2|/2q^2(\sqrt{5}+(2p-q)/q)$ $\geqq 4/2q^2(\sqrt{5}+(2p-q)/q)>0$ $(p^2-pq-q^2=(-1)^n)$
- $p/q \sim \phi \rightarrow |\phi-p/q| \geqq 1/q^2(\sqrt{5})$

Pell numbers

- $\{P_n\}=\{1,2,5,12,29,70,169,408,985,2378,,,\}$
- quasi-exponential, $\sim(1+\sqrt{2})^n$

- $f(p/q)=(p+2q)/(p+q)$ (Diophantine approximation)
- $p_{n+1}=p_n+2q_n$, $q_{n+1}=p_n+q_n$
- $1/1,3/2,7/5,17/12,41/29,99/70,239/169,,, \rightarrow \sqrt{2}$
- Denominator of best approximating fraction
- $p(p+q)-(p+2q)q=p^2-2q^2=(-1)^n$ (Pell equation)
- $p^2-2q^2\neq0$, if $p^2-2q^2=0$ then $\sqrt{2}$ is rational.
- $1/1<7/5<41/29<239/169<<\sqrt{2}<<99/70<17/12<3/2$

Pell numbers

- $\{P_n\}=\{1,2,5,12,29,70,169,408,985,2378,,\}$
- quasi-exponential, $\sim(1+\sqrt{2})^n$

- f(p/q)=(p+2q)/(p+q) (Diophantine approximation)
- $p_{n+1}=p_n+2q_n$, $q_{n+1}=p_n+q_n$
- 1/1,3/2,7/5,17/12,41/29,99/70,239/169,,, $\rightarrow \sqrt{2}$
- $|\sqrt{2}-99/70| \sim 7\text{x}10^{-5}$
- $|\sqrt{2}-47321/33461| \sim 3\text{x}10^{-10}$
- Error size is smaller than ϕ

λ (Lagrange number) is a index of Diophantine approximation

- $|\sqrt{2}-p/q|<1/\lambda(q)^2$, $\lambda=\sqrt{8}$
- Slow convergence, not good precision
- $|\sqrt{2}-1/1|$~0.414
- $|\sqrt{2}-3/2|$~0.086
- $|\sqrt{2}-7/5|$~0.014
- $|\sqrt{2}-17/12|$~0.002
- $|\sqrt{2}-41/29|$~0.0004
- $|\sqrt{2}-99/70|$~0.00007
- $|\sqrt{2}-239/169|$~0.00001

- $|\phi-p/q|<1/\lambda(q)^2$, $\lambda=\sqrt{5}$
- Slowest convergence, poor precision
- $|\phi-1/1|$~0.618
- $|\phi-2/1|$~0.382
- $|\phi-3/2|$~0.118
- $|\phi-5/3|$~0.049
- $|\phi-8/5|$~0.018
- $|\phi-13/8|$~0.007
- $|\phi-21/13|$~0.0026

$\lambda=\sqrt{8}$ is not possible to decrease any more.

- Upper bound
- $|\sqrt{2}-p/q|<1/\lambda(q)^2$, $\lambda=\sqrt{8}$ holds for infinitely many p/q.
- 部分商がすべて2
- $\alpha=\sqrt{2} \to \lambda=[2;2,2,2,,,]+[0;2,2,2,,,]=1+\sqrt{2}-1+\sqrt{2}=\sqrt{8}$

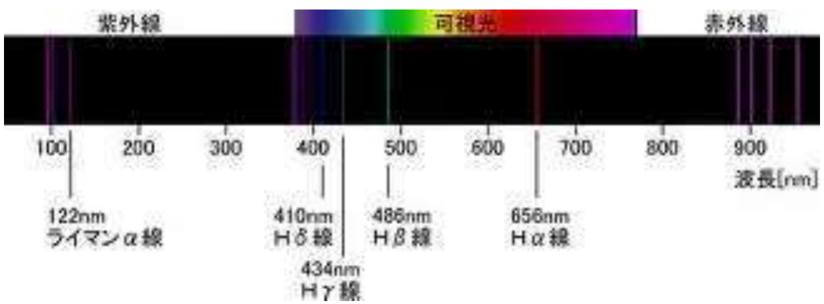
- Lower bound
- $|\sqrt{2}-p/q|=|2q^2-p^2|/q^2(\sqrt{2}+p/q) \geqq 1/q^2(\sqrt{2}+p/q)>0$
- $p/q \sim \sqrt{2} \to |\sqrt{2}-p/q| \geqq 1/q^2(\sqrt{8})$ $(p^2-2q^2=(-1)^n)$

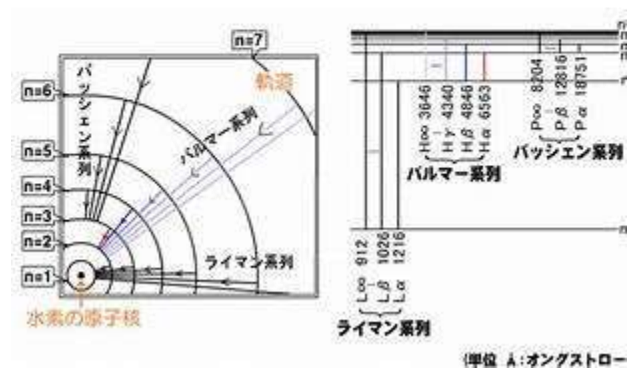
Lagrange spectrum

- $|\alpha-p/q|<1/\lambda(q)^2$ (Diophantine approximation)

- x=1,2,5,13,29,34,89,169,194,233,433,610,985,,,
- $\lambda=(9x^2-4)^{1/2}/x$, $[\sqrt{5},3]$
- $\sqrt{5}/1<\sqrt{8}=\sqrt{32}/2<\sqrt{221}/5<\sqrt{1517}/13<\sqrt{7565}/29$ $<\ldots<3$
- 離散的
- 部分商がすべて1<すべて2<すべて1または2

Hydrogen spectrum, 離散スペクトル





m=2(バルマー系列)

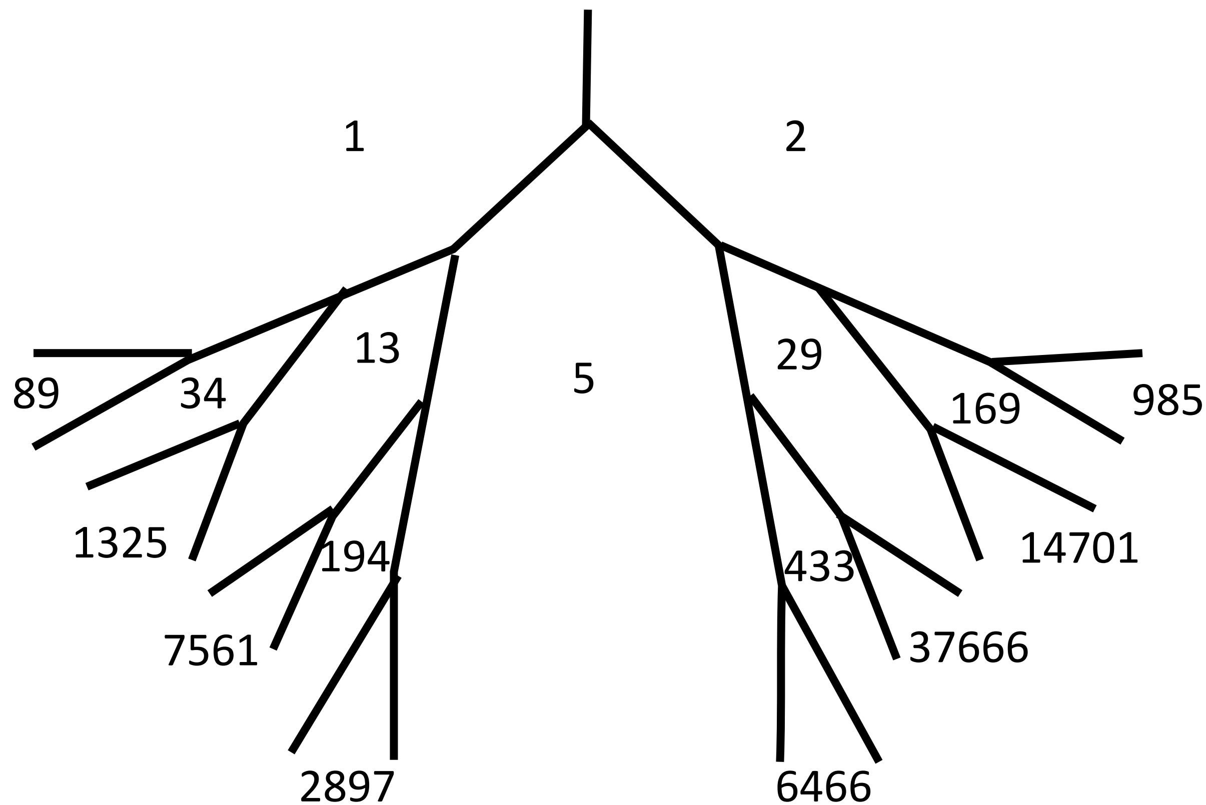
$1/\lambda=R(1/m^2-1/n^2)$ $\lambda/\lambda_0=(n^2-m^2)/n^2$	$m=2, \lambda_0=365$ $\lambda/\lambda_0=(n^2-4)/n^2$
m=1 (ライマン)	$n=3$, $\lambda=656$
m=2 (バルマー)	$n=4$, $\lambda=486$
m=3 (パッシェン)	$n=5$, $\lambda=434$
m=4 (ブラケット)	$n=6$, $\lambda=410$
m=5 (プント)	$n=7$, $\lambda=397$
m=6 (ハンフリーズ)	$\lambda=(9x^2-4)^{1/2}/x$

Markov numbers

- $|\alpha - p/q| < 1/\lambda(q)^2$ (Diophantine approximation)

- x=1,2,5,13,29,34,89,169,194,233,433,610,985,,,
- $\lambda = (9x^2-4)^{1/2}/x$, $[\sqrt{5}, 3]$
- $\sqrt{5}/1 < \sqrt{8} = \sqrt{32}/2 < \sqrt{221}/5 < \sqrt{1517}/13 < \sqrt{7565}/29$ $< \ldots < 3$
- Denominator of λ (Lagrange number)

Markov tree (topograph)

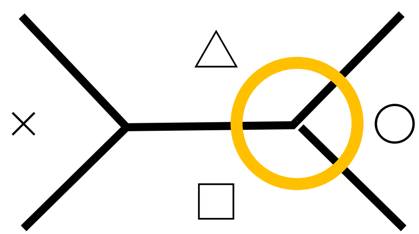


Markov numbers

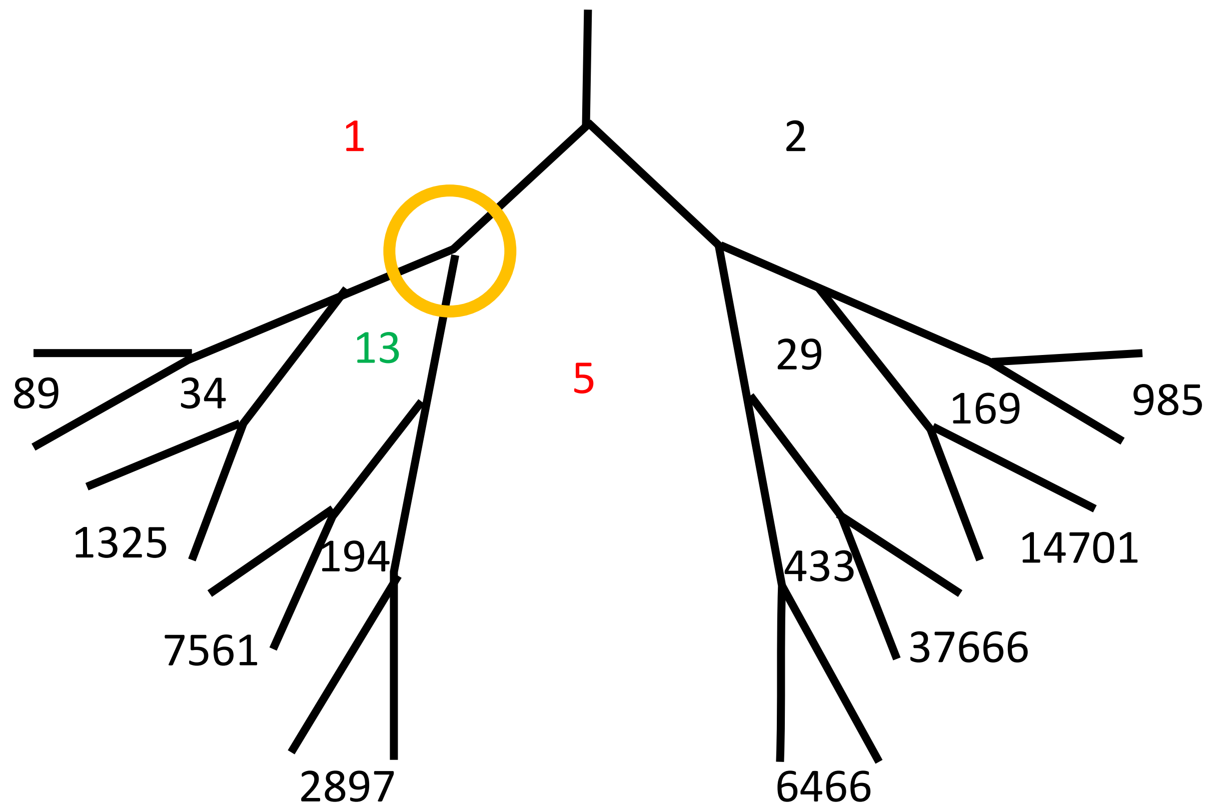
- $x^2+y^2+z^2=3xyz$ (Markov equation)
- $(1,2,5) \rightarrow 1+4+25=3x1x2x5=30$
- $(1,5,13)=1+25+169=3x1x5x13=195$
- $(2,5,29)=4+25+841=3x2x5x29=870$

- Markov equation is equivalent to;
- $f(x,y,z)=x^2+y^2+z^2-3xyz=0$
- $f(x,y,3xy-z)=x^2+y^2+z^2-3xyz=0$

$\bigcirc^2+\triangle^2+\square^2=3\bigcirc\triangle\square$ for Markov



$1^2+5^2+13^2=3\text{x}1\text{x}5\text{x}13$ (around the vertex)



Markov numbers

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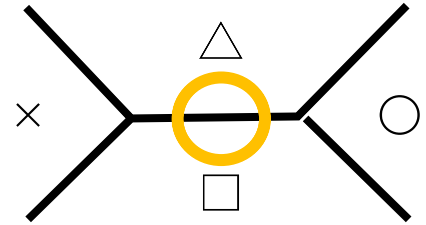
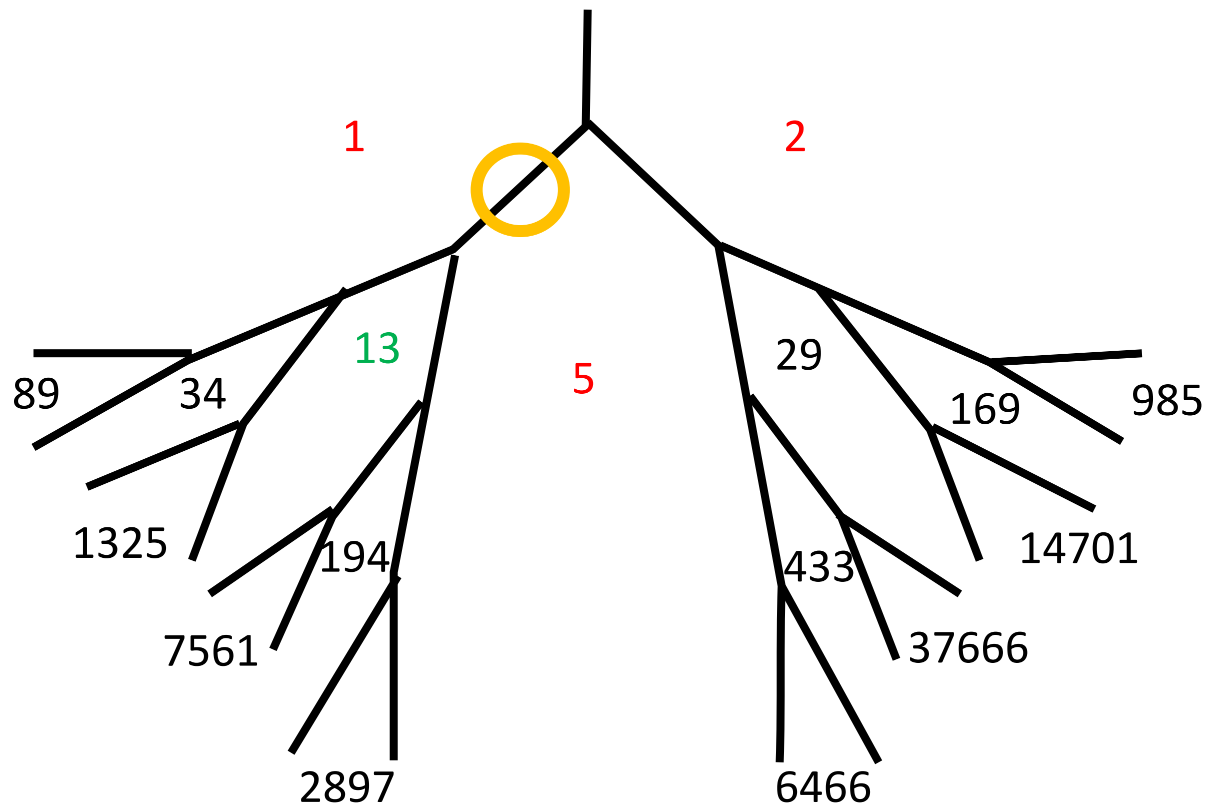
- Markov equation is equivalent to;
- $f(x,y,z)=x^2+y^2+z^2-3xyz=0$
- $f(x,y,3xy-z)=x^2+y^2+z^2-3xyz=0$, easy to calculate

○+×=3△□ scheme for Markov



2+13=3x1x5 (around the edge)

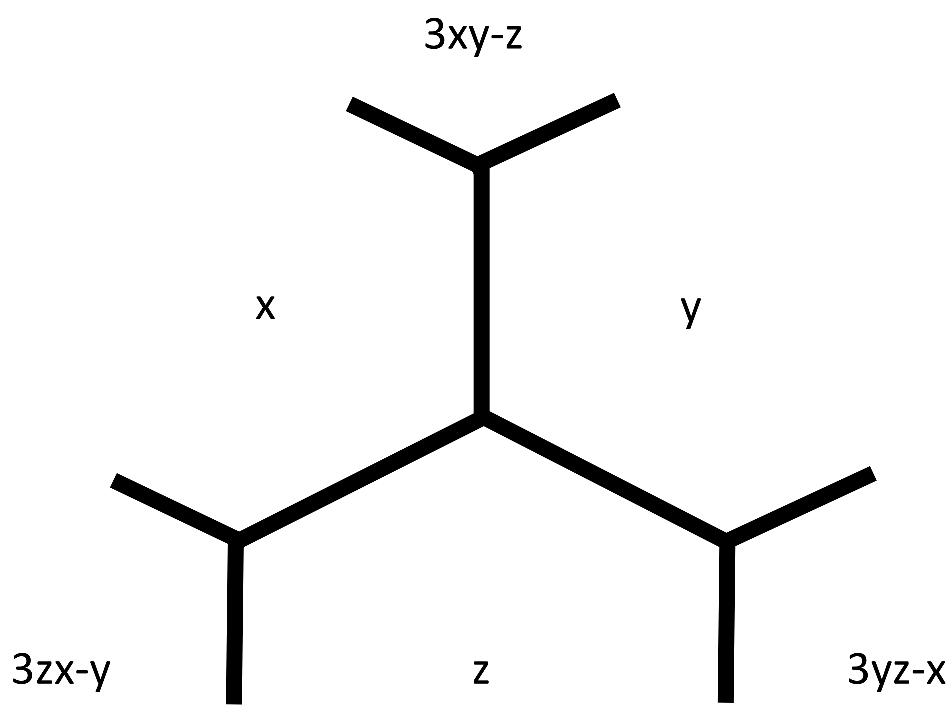


Markov numbers

- $x^2+y^2+z^2=3xyz$ (Markov equation)
- $(1,2,5) \rightarrow 1+4+25=3\times1\times2\times5=30$
- $(1,5,13)=1+25+169=3\times1\times5\times13=195$
- $(2,5,29)=4+25+841=3\times2\times5\times29=870$

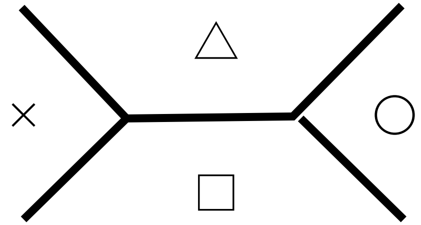
- $f(x,y,z)=x^2+y^2+z^2-3xyz=0$
- $f(x,y,3xy-z)=x^2+y^2+z^2-3xyz=0$
- $f(x,3zx-y,z)=x^2+y^2+z^2-3xyz=0$
- $f(3yz-x,y,z)=x^2+y^2+z^2-3xyz=0$

Three neighbors of Markov triplet

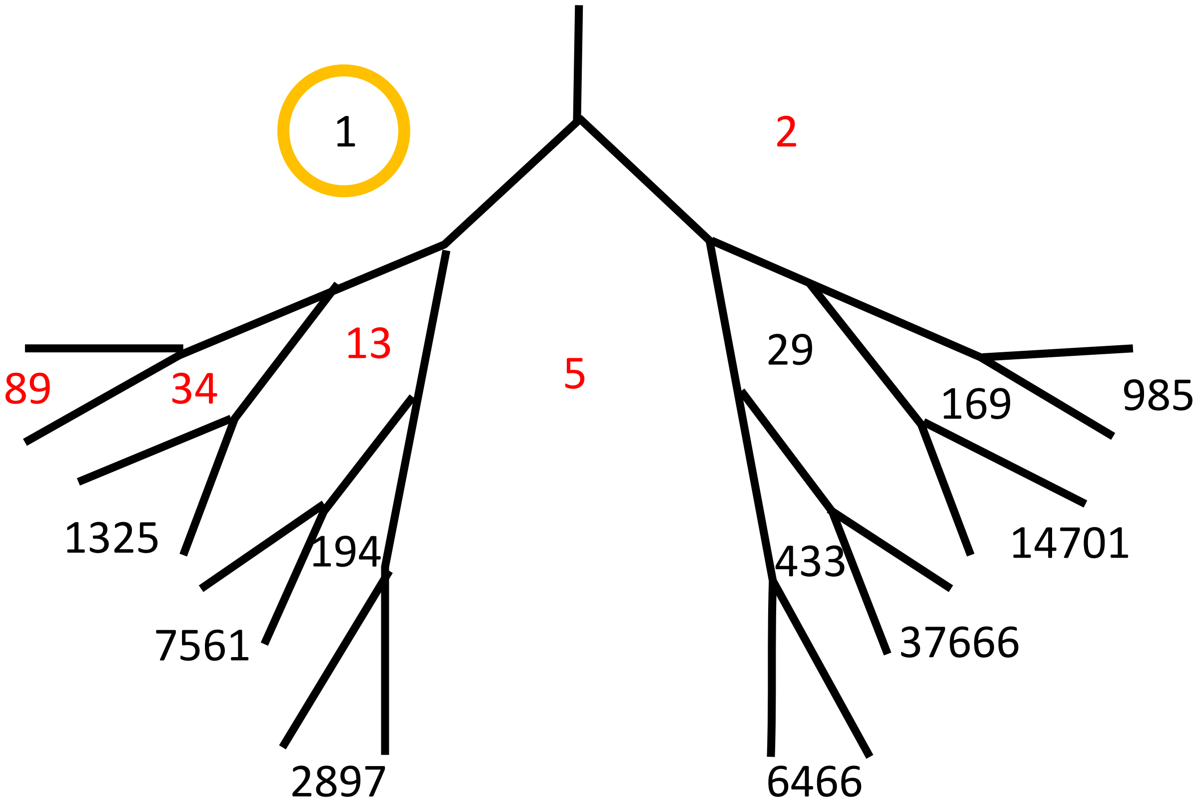


○+× =3△□ scheme for Markov

- △=1 → $u_{n+1}=3u_n-u_{n-1}$ (Fibonacci)
- $F_{2n-1}^2-3F_{2n-1}F_{2n+1}+F_{2n+1}^2=-1$, $F_{2n+1}=F_n^2+F_{n+1}^2$

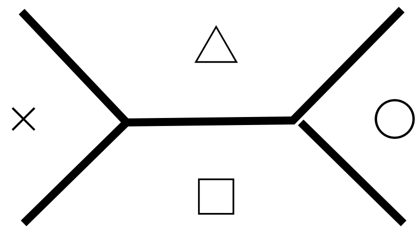


Fibonacci numbers in Markov tree

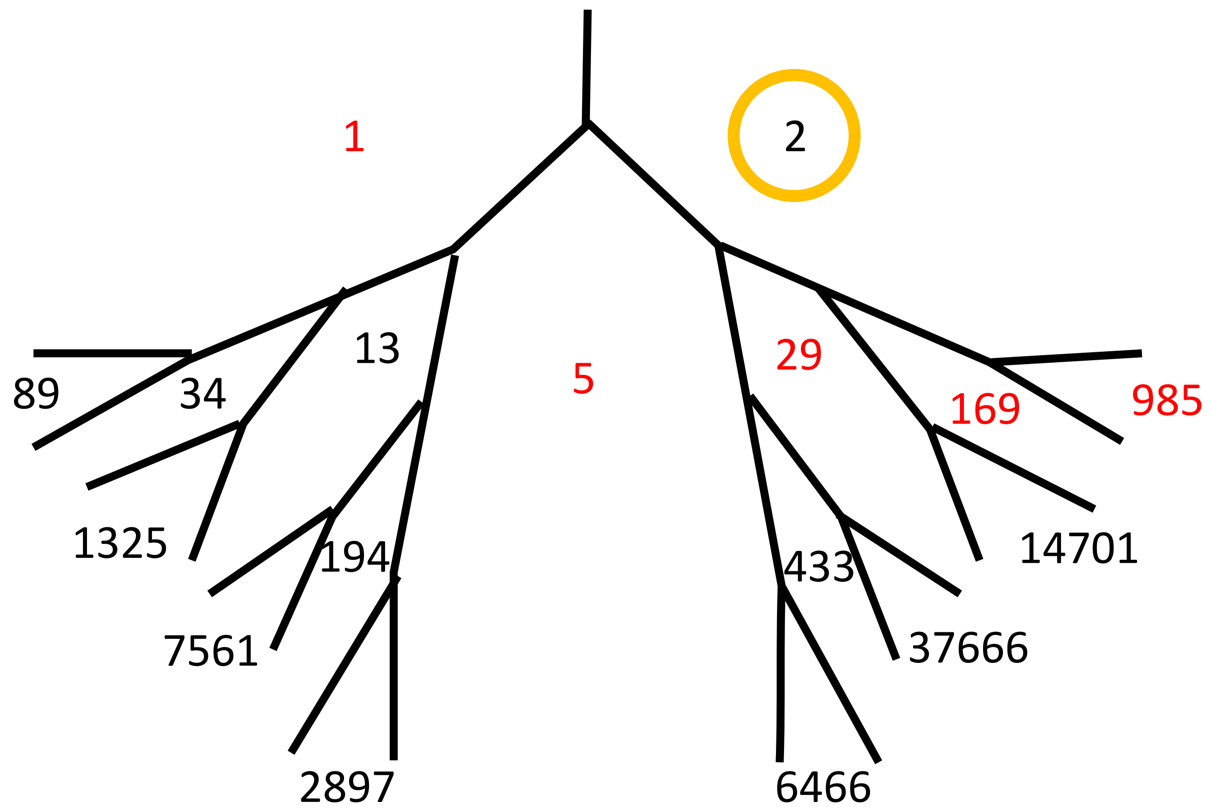


○+ × =3△□ scheme for Markov

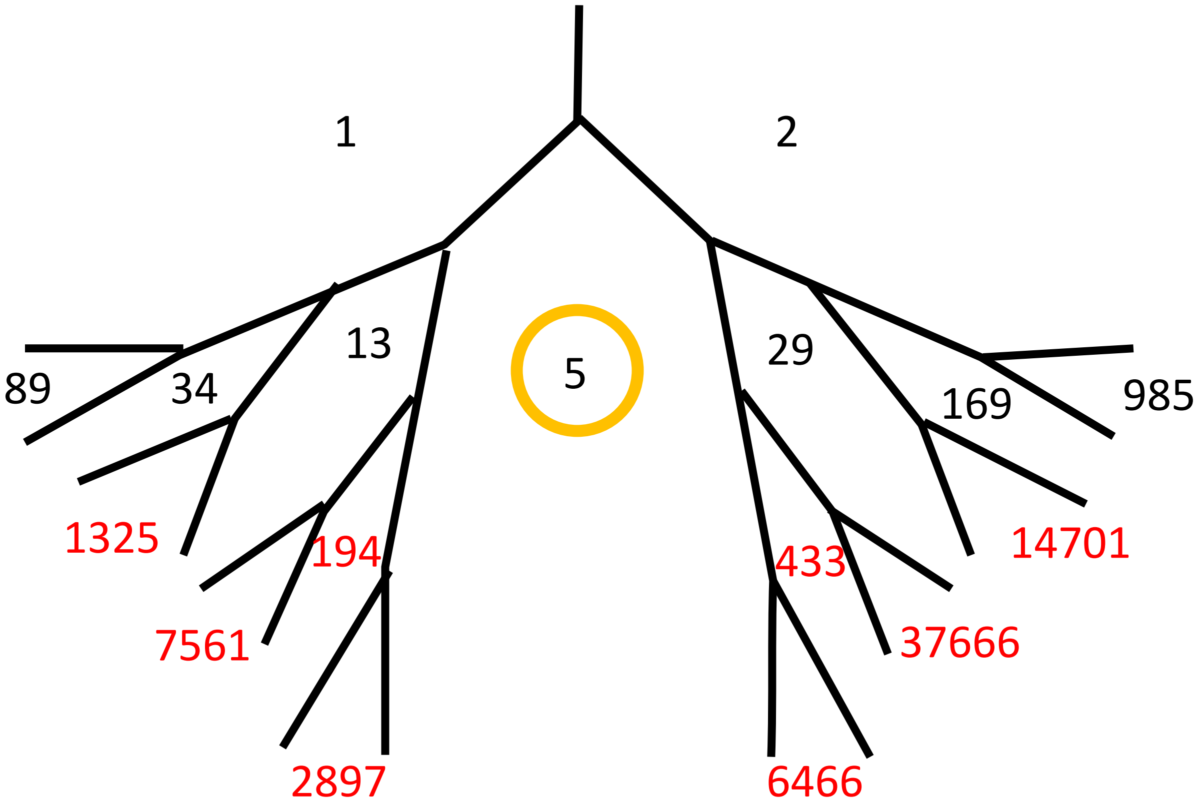
- △=2 → $u_{n+1}=6u_n-u_{n-1}$ (Pell)
- $P_{2n-1}{}^2-6P_{2n-1}P_{2n+1}+P_{2n+1}{}^2=-4$, $z^2=x^2+(x+1)^2$



Pell numbers in Markov tree



Neither Fibonacci nor Pell



Markov numbers

- $x^2+y^2+z^2=3xyz$ (not quadratic form)
- $(1,2,5) \rightarrow 1+4+25=3\times1\times2\times5=30$
- $(1,5,13)=1+25+169=3\times1\times5\times13=195$
- $(2,5,29)=4+25+841=3\times2\times5\times29=870$

- 1,2,5,13,29,34,89,169,194,233,433,610,985,,,
- 1,2,5,13,4,89,233,610: $\{F_{2n-1}\}$
- 1,5,29,169,985,5741: $\{P_{2n-1}\}$
- 1,5: common for $\{F_{2n-1}\}$ and $\{P_{2n-1}\}$
- 194,433,1325,2897: non$\{F_{2n-1}\}$ non$\{P_{2n-1}\}$

Markov numbers

Markov numbers, x^2+y^2+z^2=3xyz										
M[n]	**1**	**2**	**5**	**13**	**29**	**34**	**89**	**169**	**194**	**233**
F[2n-1]	**1**	2	**5**	13		34	89			233
P[2n-1]	**1**		**5**		29			169		
M[n]	**433**	**610**	**985**	**1325**	**1597**	**2897**	**4181**	**5741**	**6466**	**7561**
F[2n-1]		610			1597		4181			
P[2n-1]			985					5741		

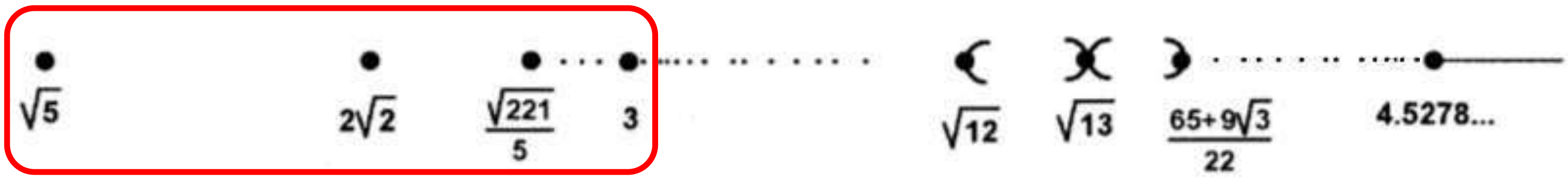
Markov Theorem, $\lambda=[\sqrt{5},3]$

- $|\alpha-p/q|<1/\lambda(q)^2$ (Diophantine approximation)

- x=1,2,5,13,29,34,89,169,194,233,433,610,985,,,
- $\lambda=(9x^2-4)^{1/2}/x$, $[\sqrt{5},3]$
- $\sqrt{5}/1<\sqrt{8}=\sqrt{32}/2<\sqrt{221}/5<\sqrt{1517}/13<\sqrt{7565}/29<\ldots<3$

- Denominator of λ (Lagrange number);
- $\{F_{2n-1}\}$, $\{P_{2n-1}\}$, non$\{F_{2n-1}\}$ non$\{P_{2n-1}\}$
- 1 and 5 overlaps between $\{F_{2n-1}\}$ and $\{P_{2n-1}\}$

Markov Lagrange spectrum



- α→λ=[3;2,1,2,1,,,]+[0;3,2,1,2,1,,,]
  →$(55+11\sqrt{3})/22+(10-2\sqrt{3})/22=(65+9\sqrt{3})/22\sim 3.6631$

- Perron's gap: maximal open interval
  $(\sqrt{12}, \sqrt{13})$, $(\sqrt{13}, (65+9\sqrt{3})/22)$
- Hall's ray: everywhere dense, Freiman constant
  $\lambda>(2221564096+283748\sqrt{462})/491993569\sim 4.5278$
- Almost decreasing gap size ($\lambda=\sqrt{5}\to\sqrt{8}\to\sqrt{221}/5\to\to\to 3$)

λ (Lagrange number) is a index of Diophantine approximation

- $\alpha=[a_0;a_1,a_2,a_3,,,]$, $|\alpha-p/q|<1/\lambda(q)^2$
- $\lambda=[a_{n+1};a_{n+2},a_{n+3},,,]+[0;a_n,a_{n-1},,,a_1]$

- $\{F_{2n-1}\}$
- $\alpha\rightarrow\lambda=[2;1,1,2,,,]+[0;2,1,1,2,,,]$

  $\rightarrow(11+\sqrt{221})/10+(-11+\sqrt{221})/10=\sqrt{221}/5$
- $\alpha\rightarrow\lambda=[2;1,1,1,1,2,,,]+[0;2,1,1,1,1,2,,,]$

  $\rightarrow(29+\sqrt{1517})/26+(-29+\sqrt{1517})/26=\sqrt{1517}/13$

λ (Lagrange number) is a index of Diophantine approximation

- $\alpha=[a_0;a_1,a_2,a_3,,,]$, $|\alpha-p/q|<1/\lambda(q)^2$
- $\lambda=[a_{n+1};a_{n+2},a_{n+3},,,]+[0;a_n,a_{n-1},,,a_1]$

- $\{P_{2n-1}\}$
- $\alpha\rightarrow\lambda=[2;1,1,2,2,2,,,]+[0;2,2,2,1,1,2,,,]$

  $\rightarrow(63+\sqrt{7565})/58+(-63+\sqrt{7565})/58=\sqrt{7565}/29$
- $\alpha\rightarrow\lambda=[2;1,1,2,2,2,2,2,,,]+[0;2,2,2,2,2,1,1,2,,,]$

  $\rightarrow(367+\sqrt{257045})/338+(367+\sqrt{257045})/338$

  $=\sqrt{257045}/169$

λ (Lagrange number) is a index of Diophantine approximation

- $\alpha=[a_0;a_1,a_2,a_3,,,]$, $|\alpha-p/q|<1/\lambda(q)^2$
- $\lambda=[a_{n+1};a_{n+2},a_{n+3},,,]+[0;a_n,a_{n-1},,,a_1]$

- Non$\{F_{2n-1}\}$ non$\{P_{2n-1}\}$
- $\alpha\rightarrow\lambda=[2;1,1,1,1,2,2,1,1,2,,]+[0;2,1,1,2,2,1,1,1,1,2,,]$

  $\rightarrow(108+\sqrt{21170})/97+(-108+\sqrt{21170})/97$

  $=\sqrt{338720}/194$

λ (Lagrange number) is a index of Diophantine approximation

- $\alpha=[a_0;a_1,a_2,a_3,,,]$, $|\alpha-p/q|<1/\lambda(q)^2$
- $\lambda=[a_{n+1};a_{n+2},a_{n+3},,,]+[0;a_n,a_{n-1},,,a_1]$

- λ, excluded from Markov numbers
- $\alpha\rightarrow\lambda=[2;1,1,2,1,2,,,]+[0;2,1,1,2,1,2,,,]$

  $\rightarrow(6+\sqrt{3})/3+3/(6+\sqrt{3})=(84+8\sqrt{3})/33\sim2.965$

λ (Lagrange number) is a index of Diophantine approximation

- $\alpha=[a_0;a_1,a_2,a_3,,,]$, $|\alpha-p/q|<1/\lambda(q)^2$
- $\lambda=[a_{n+1};a_{n+2},a_{n+3},,,]+[0;a_n,a_{n-1},,,a_1]$

- λ, excluded from Markov numbers
- $\alpha\rightarrow\lambda=[2;1,1,1,1,1,,,]+[0;2,1,1,1,1,1,,,]$
  $\rightarrow\phi+1+1/(\phi+1)=3$
- $\alpha\rightarrow\lambda=[2;2,1,1,1,,,]+[0;1,1,1,,,]$
  $\rightarrow 4-\phi+\phi-1=3$
- $\alpha\rightarrow\lambda=[2;1,1,2,2,2,,,]+[0;2,2,2,,,]$
  $\rightarrow 4-\sqrt{2}+\sqrt{2}-1=3$

Indefinite binary quadratic form

Markov number	ax^2	bxy	cy^2
1 (F)	1	-1	-1
2 (F)	2	-4	-2
5 (F)	5	-11	-5
13 (F)	13	-29	-13
29 (P)	29	-63	-31
34 (F)	34	-76	-34
89 (F)	89	-199	-89
169 (P)	169	-367	-181
194 (NFNP)	194	-432	-196
233 (F)	233	-521	-233

Now on setting problem

- Quadratic form of Fibonacci numbers
- Topographic form of Fibonacci numbers
- Topographic form of Pell numbers
- Topographic form of Markov numbers
- Quadratic form of Markov numbers
- Quadratic form of expanded Markov numbers
- Topographic form of expanded Markov numbers

Markov Theorem(1879)に欠けているものを補完

(λ>3)領域

部分商がすべて1または2の連分数α

Simple question for Markov theorem

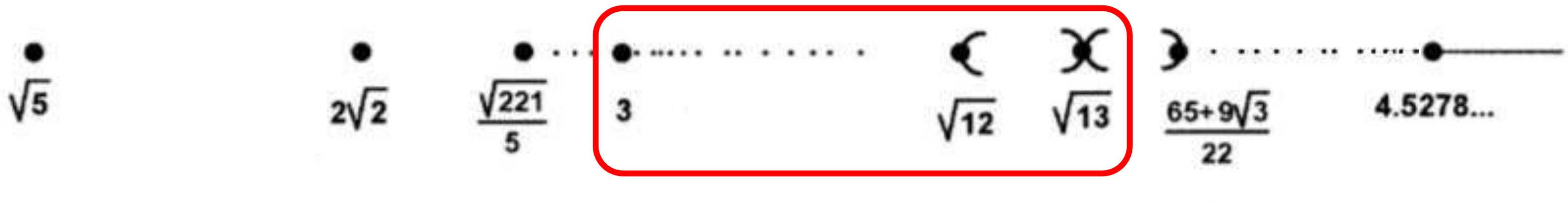
- Where is $\{F_{2n}\}$?
- Where is $\{P_{2n}\}$?

Simple question for Markov theorem

- Where is $\{F_{2n}\}$?
- Where is $\{P_{2n}\}$?
- I speculated they would appear in $\lambda=[3,\sqrt{13}]$.

- Focusing on $\lambda=[3,\sqrt{13}]$ interval
- Where is λ whose partial quotients of continued fraction are all [1,2] ?
- How are Markov equation and tree of $\lambda=[3,\sqrt{13}]$?

Markov Lagrange spectrum



- α→λ=[3;2,1,2,1,,,]+[0;3,2,1,2,1,,,]
  →$(55+11\sqrt{3})/22+(10-2\sqrt{3})/22=(65+9\sqrt{3})/22$～3.6631

- Perron's gap: maximal open interval
  $(\sqrt{12}, \sqrt{13})$, $(\sqrt{13}, (65+9\sqrt{3})/22)$
- Hall's ray: everywhere dense, Freiman constant
  $\lambda>(2221564096+283748\sqrt{462})\ /491993569$～4.5278
- Almost increasing gap size (λ=3→→$\sqrt{10}$→$\sqrt{12}$→$\sqrt{13}$)

λ (Lagrange number) is a index of Diophantine approximation

- $\alpha=[a_0;a_1,a_2,a_3,,,]$, $|\alpha-p/q|<1/\lambda(q)^2$
- $\lambda=[a_{n+1};a_{n+2},a_{n+3},,,]+[0;a_n,a_{n-1},,,a_1]$

- $\alpha=\phi$(黄金比)$\rightarrow\lambda=[1;1,1,1,,,]+[0;1,1,1,1,,,]$
  $\rightarrow\phi+1/\phi=\sqrt{5}$
- $\alpha=1+\sqrt{2}$(白銀比)$\rightarrow\lambda=[2;2,2,2,,,]+[0;2,2,2,2,,,]$
  $\rightarrow 1+\sqrt{2}-1+\sqrt{2}=\sqrt{8}$
- $\alpha=(3+\sqrt{13})/2$(青銅比)$\rightarrow\lambda=[3;3,3,3,,,]+[0;3,3,3,3,,,]$
  $\rightarrow(3+\sqrt{13})/2+(-3+\sqrt{13})/2=\sqrt{13}$

λ (Lagrange number) is a index of Diophantine approximation

- $\alpha=[a_0;a_1,a_2,a_3,,,]$, $|\alpha-p/q|<1/\lambda(q)^2$
- $\lambda=[a_{n+1};a_{n+2},a_{n+3},,,]+[0;a_n,a_{n-1},,,a_1]$

- $\{F_{2n}\}$
- $\alpha \rightarrow \lambda=[2;1,2,2,1,2,,,]+[0;2,1,2,2,1,2,,,]$
  $\rightarrow(7+\sqrt{85})/6+(-7+\sqrt{85})/6=\sqrt{85}/3$
- $\alpha \rightarrow \lambda=[2;1,1,1,2,2,1,1,1,2,,,]+[0;2,1,1,1,2,2,1,1,1,2,,,]$
  $\rightarrow(18+\sqrt{580})/16+(-18+\sqrt{580})/16=\sqrt{580}/8$
- $\alpha \rightarrow \lambda=[2;1,1,1,1,1,2,,,]+[0;2,1,1,1,1,1,2,,,]$
  $\rightarrow(47+\sqrt{3973})/42+(-47+\sqrt{3973})/42=\sqrt{3973}/21$

λ (Lagrange number) is a index of Diophantine approximation

- $\alpha=[a_0;a_1,a_2,a_3,,,]$, $|\alpha-p/q|<1/\lambda(q)^2$
- $\lambda=[a_{n+1};a_{n+2},a_{n+3},,,]+[0;a_n,a_{n-1},,,a_1]$

- $\{P_{2n}\}$
- $\alpha\rightarrow\lambda=[2;1,1,2,2,,,,]+[0;2,2,1,1,2,,,]$

  $\rightarrow(13+\sqrt{325})/12+(-13+\sqrt{325})/12=\sqrt{1300}/12$
- $\alpha\rightarrow\lambda=[2;1,1,2,2,2,2,,,]+[0;2,2,2,2,1,1,2,,,]$

  $\rightarrow(76+\sqrt{11026})/70+(-76+\sqrt{11026})/70$

  $=\sqrt{44104}/70$

λ (Lagrange number) is a index of Diophantine approximation

- $\alpha=[a_0;a_1,a_2,a_3,,,]$, $|\alpha-p/q|<1/\lambda(q)^2$
- $\lambda=[a_{n+1};a_{n+2},a_{n+3},,,]+[0;a_n,a_{n-1},,,a_1]$

- λ, excluded from Markov numbers
- $\alpha\to\lambda=[2;1,2,1,,,]+[0;2,2,2,1,2,1,,,]$

  $\to(1+\sqrt{3})+(1-\sqrt{3}/3)=(6+2\sqrt{3})/3\sim3.154$
- $\alpha\to\lambda=[2;2,2,1,2,1,,,]+[0;1,2,1,2,,,]$

  $\to(-1+\sqrt{3})+(3-\sqrt{3}/3)=(6+2\sqrt{3})/3\sim3.154$

λ (Lagrange number) is a index of Diophantine approximation

- $\alpha=[a_0;a_1,a_2,a_3,,,]$, $|\alpha-p/q|<1/\lambda(q)^2$
- $\lambda=[a_{n+1};a_{n+2},a_{n+3},,,]+[0;a_n,a_{n-1},,,a_1]$
- $\alpha=[n;n,n,n,n,n,n,,,]=\{n+(n^2+4)^{1/2}\}/2$

  $\rightarrow\lambda=[n;n,n,n,,,]+[0;n,n,n,n,,,]\rightarrow(n^2+4)^{1/2}$
- $\alpha=[n;1,n,1,n,1,n,,,]=\{n+(n^2+4n)^{1/2}\}/2$

  $\rightarrow\lambda=[n;1,n,1,,,]+[0;1,n,1,n,,,]\rightarrow(n^2+4n)^{1/2}$
- λ, excluded from Markov numbers
- $\alpha=\sqrt{3}=[1;1,2,1,2,,,]\rightarrow\lambda=[2;1,2,1,2,,]+[0;1,2,1,2,,,]$

  $\rightarrow 1+\sqrt{3}-1+\sqrt{3}=2\sqrt{3}=\sqrt{12}$

Expanded Markov numbers, $\lambda=[3,\sqrt{13}]$

- $|\alpha-p/q|<1/\lambda(q)^2$ (Diophantine approximation)

- x=1,2,3,8,12,21,55,70,144,377,408,987,,,
- $\lambda=(9x^2+4)^{1/2}/x$, $[3,\sqrt{13}]$
- $3<\ldots<\sqrt{44104}/70<\sqrt{27229}/55<\sqrt{3973}/21<$ $<\sqrt{1300}/12<\sqrt{580}/8<\sqrt{85}/3<\sqrt{40}/2<\sqrt{13}/1$
- 部分商がすべて1または2<すべて3

- Denominator of λ; $\{F_{2n}\}$, $\{P_{2n}\}$
- No overlaps between $\{F_{2n}\}$ and $\{P_{2n}\}$

Indefinite binary quadratic form

Markov number	ax^2	bxy	cy^2
1 (F)	1	-3	-1
2 (P)	2	-4	-3
3 (F)	3	-7	-3
8 (F)	8	-12	-8
12 (P)	12	-26	-13
21 (F)	21	-47	-21
55 (F)	55	-123	-55
70 (P)	70	-152	-75
144 (F)	144	-322	-144
377 (F)	377	-843	-377

Expanded Markov numbers, $\lambda=[3,\sqrt{13}]$

- $|\alpha - p/q| < 1/\lambda(q)^2$ (Diophantine approximation)

- $x=1,2,3,8,12,21,55,70,144,377,408,987,,,$
- $\lambda=(9x^2+4)^{1/2}/x$, $[3,\sqrt{13}]$
- $3<\ldots<\sqrt{44104}/70<\sqrt{27229}/55<\sqrt{3973}/21<$
  $<\sqrt{1300}/12<\sqrt{580}/8<\sqrt{85}/3<\sqrt{40}/2<\sqrt{13}/1$

- Markov equation of $\lambda=[3,\sqrt{13}]$?
- Markov tree of $\lambda=[3,\sqrt{13}]$?

Plentiful harvest of Diophantine equation

- $(x^2+y^2+1)/xy=3 \leftarrow\rightarrow x=F_{2n-1}, y=F_{2n+1}$
- $(x^2+y^2-1)/xy=3 \leftarrow\rightarrow x=F_{2n}, y=F_{2n+2}$
- $(x^2+y^2-5)/xy=3 \leftarrow\rightarrow x=L_{2n-1}, y=L_{2n+1}$
- $(x^2+y^2+5)/xy=3 \leftarrow\rightarrow x=L_{2n}, y=L_{2n+2}$
- $(x^2+y^2+4)/xy=6 \leftarrow\rightarrow x=P_{2n-1}, y=P_{2n+1}$
- $(x^2+y^2-4)/xy=6 \leftarrow\rightarrow x=P_{2n}, y=P_{2n+2}$
- $(x^2+y^2-32)/xy=6 \leftarrow\rightarrow x=Q_{2n-1}, y=Q_{2n+1}$
- $(x^2+y^2+32)/xy=6 \leftarrow\rightarrow x=Q_{2n}, y=Q_{2n+2}$

- $\{x+y+xy(x+y)\}\{1/x+1/y+1/xy(x+y)\} =(x+y)^2+6 \leftarrow\rightarrow x=F_{2n-1}, y=F_{2n+1}$
- $\{x+y+xy(x+y)\}\{1/x+1/y+4/xy(x+y)\} =(x+y)^2+12 \leftarrow\rightarrow x=P_{2n-1}, y=P_{2n+1}$

Plentiful harvest of Diophantine equation

- $(x^2+y^2+1)/xy=3 \leftarrow\rightarrow x=F_{2n-1}, y=F_{2n+1}$
- $(x^2+y^2+4)/xy=6 \leftarrow\rightarrow x=P_{2n-1}, y=P_{2n+1}$

- $(x^2+y^2-1)/xy=3 \leftarrow\rightarrow x=F_{2n}, y=F_{2n+2}$
- $(x^2+y^2-4)/xy=6 \leftarrow\rightarrow x=P_{2n}, y=P_{2n+2}$

- $\{F_{2n}\}$ and $\{P_{2n}\}$ appear in expanded Markov number.

Markov equation

- $\lambda=[\sqrt{5},3]$
- $x^2+y^2+z^2=3xyz \leftarrow x=F_{2n-1}, y=F_{2n+1}, z=1$
- $x^2+y^2+z^2=3xyz \leftarrow x=P_{2n-1}, y=P_{2n+1}, z=2$

- $\lambda=[3,\sqrt{13}]$ (Minkowski-Markov?)
- $x^2+y^2-z^2=3xyz \leftarrow x=F_{2n}, y=F_{2n+2}, z=1$
- $x^2+y^2-z^2=3xyz \leftarrow x=P_{2n}, y=P_{2n+2}, z=2$

Three neighbors of Markov triplet

- $\lambda=[3,\sqrt{13}]$
- $f(x,y,z)=x^2+y^2-z^2-3xyz=0$ (Minkowski-Markov?)

- $f(x,y,-3xy-z)=x^2+y^2-z^2-3xyz=0$ ($-3xy-z<0$, not favorable)
- $f(x,-3zx-y,z)=x^2+y^2-z^2-3xyz=0$ ($-3zx-y<0$, not favorable)
- $f(-3yz-x,y,z)=x^2+y^2-z^2-3xyz=0$ ($-3yz-x<0$, not favorable)

Plentiful harvest of Diophantine equation

- $(x^2+y^2-1)/xy=3 \leftarrow\rightarrow x=F_{2n}, y=F_{2n+2}$
- $(x^2+y^2-4)/xy=6 \leftarrow\rightarrow x=P_{2n}, y=P_{2n+2}$

- equivalent
- $(x^2+y^2+1)/xy=3+2/xy \leftarrow\rightarrow x=F_{2n}, y=F_{2n+2}$
- $(x^2+y^2+4)/xy=6+8/xy \leftarrow\rightarrow x=P_{2n}, y=P_{2n+2}$

Markov equation

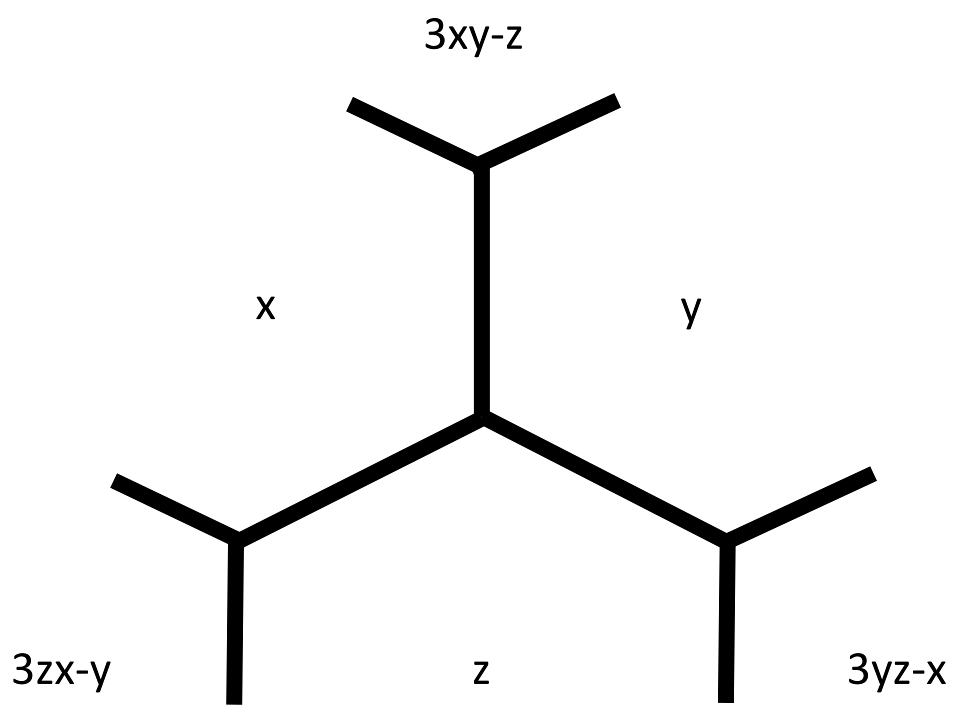
- $\lambda=[\sqrt{5},3]$
- $x^2+y^2+z^2=3xyz \leftarrow x=F_{2n-1}, y=F_{2n+1}, z=1$
- $x^2+y^2+z^2=3xyz \leftarrow x=P_{2n-1}, y=P_{2n+1}, z=2$

- $\lambda=[3,\sqrt{13}]$
- $x^2+y^2+z^2=3xyz+2 \leftarrow x=F_{2n}, y=F_{2n+2}, z=1$
- $x^2+y^2+z^2=3xyz+8 \leftarrow x=P_{2n}, y=P_{2n+2}, z=2$

Markov equation

- $\lambda=[3,\sqrt{13}]$
- $f(x,y,z)=x^2+y^2+z^2-3xyz-c=0$ for any constant c
- $f(x,y,3xy-z)=x^2+y^2+z^2-3xyz-c=0$ ($3xy-z>0$, favorable)
- $f(x,3zx-y,z)=x^2+y^2+z^2-3xyz-c=0$ ($3zx-y>0$, favorable)
- $f(3yz-x,y,z)=x^2+y^2+z^2-3xyz-c=0$ ($3yz-x>0$, favorable)

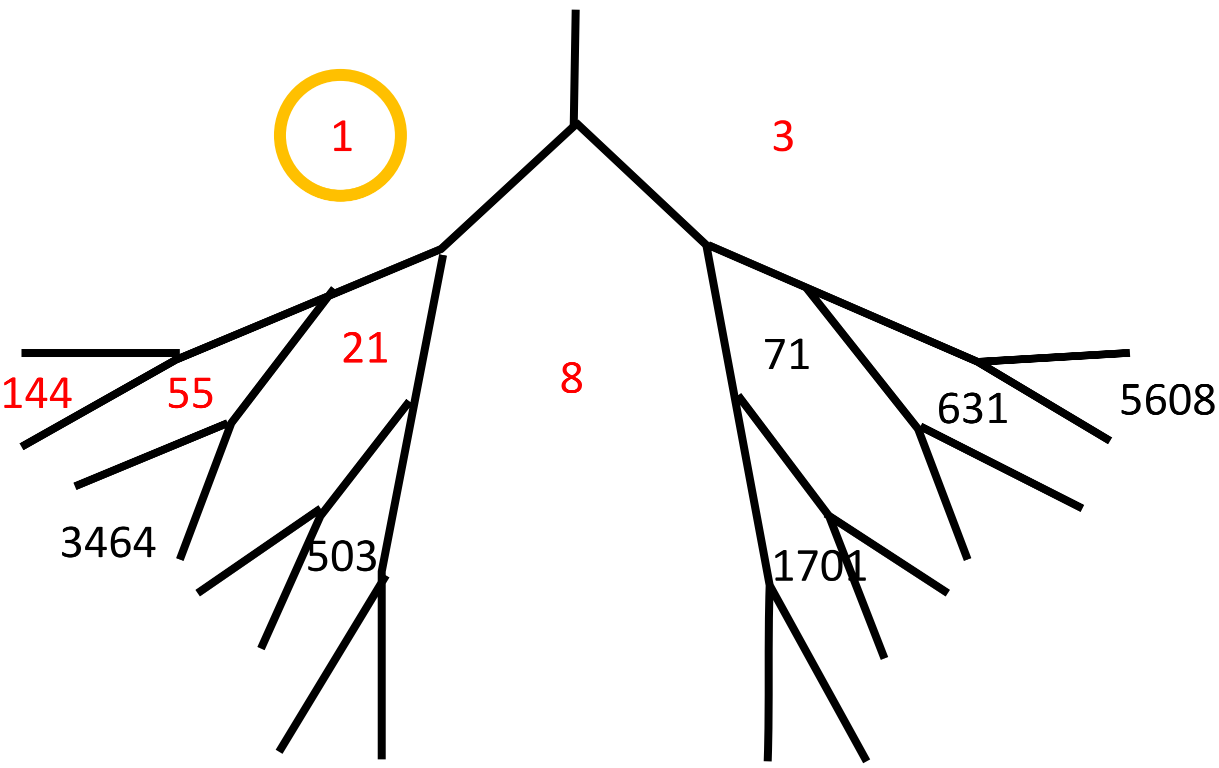
Three neighbors of Markov triplet



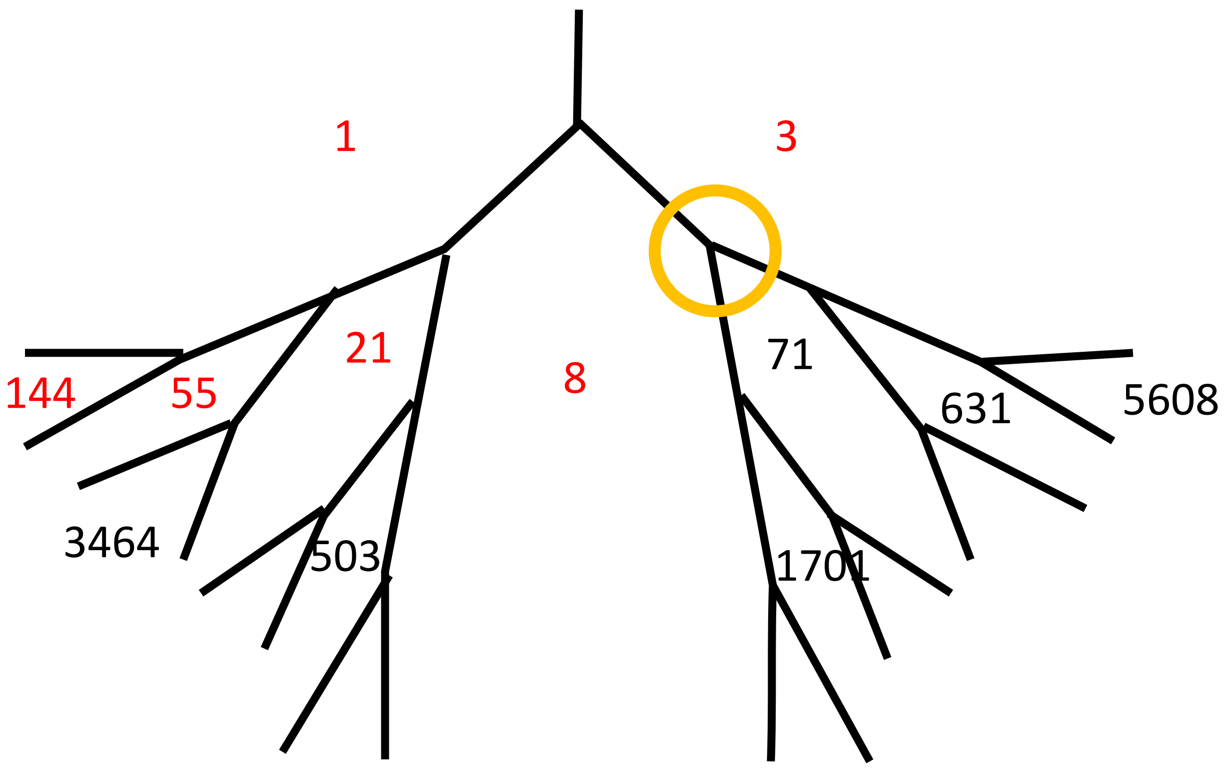
Fibonacci numbers

Fibonacci numbers: F[n+1]=F[n]+F[n-1], F1=1, F2=1										
F[n]	1	1	2	3	5	8	13	21	34	55
F[2n-1]	1		2		5		13		34	
F[2n]		**1**		**3**		**8**		**21**		**55**
F[n]	89	144	233	377	610	987	1597	2584	4181	6765
F[2n-1]	89		233		610		1597		4181	
F[2n]		**144**		**377**		**987**		**2584**		**6765**

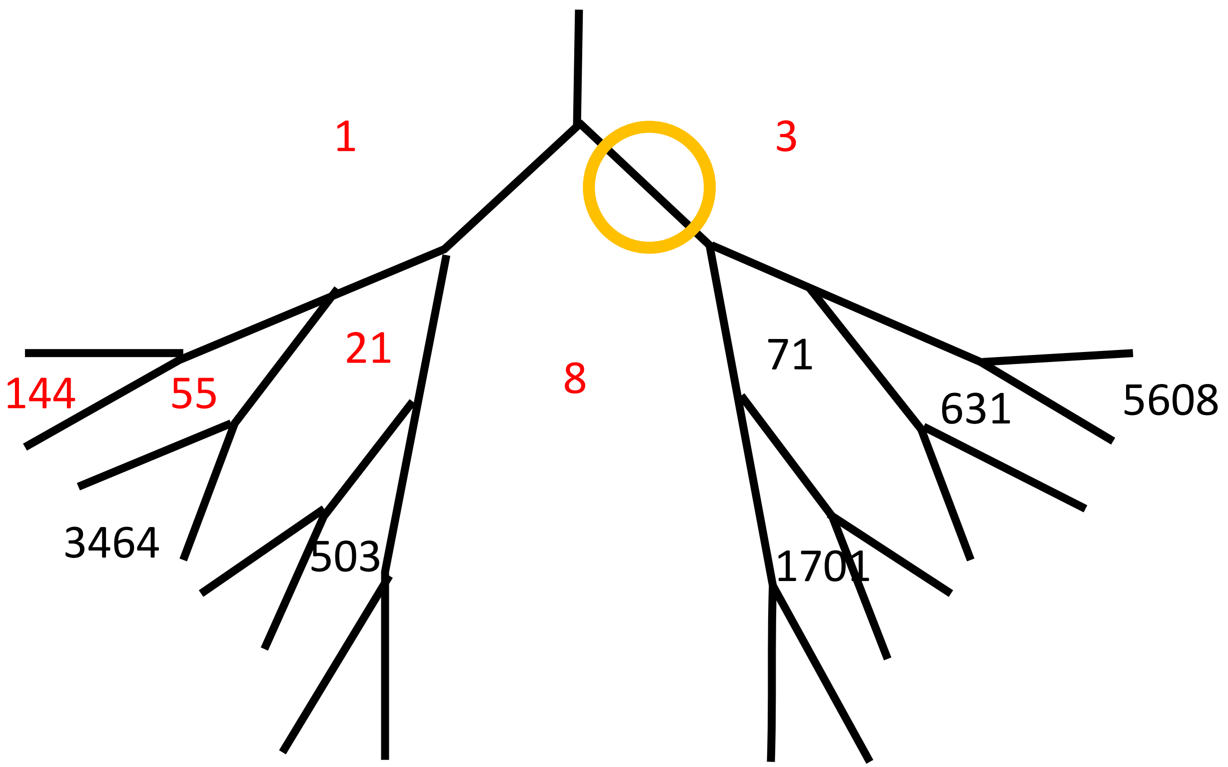
Markov tree (Fibonacci)



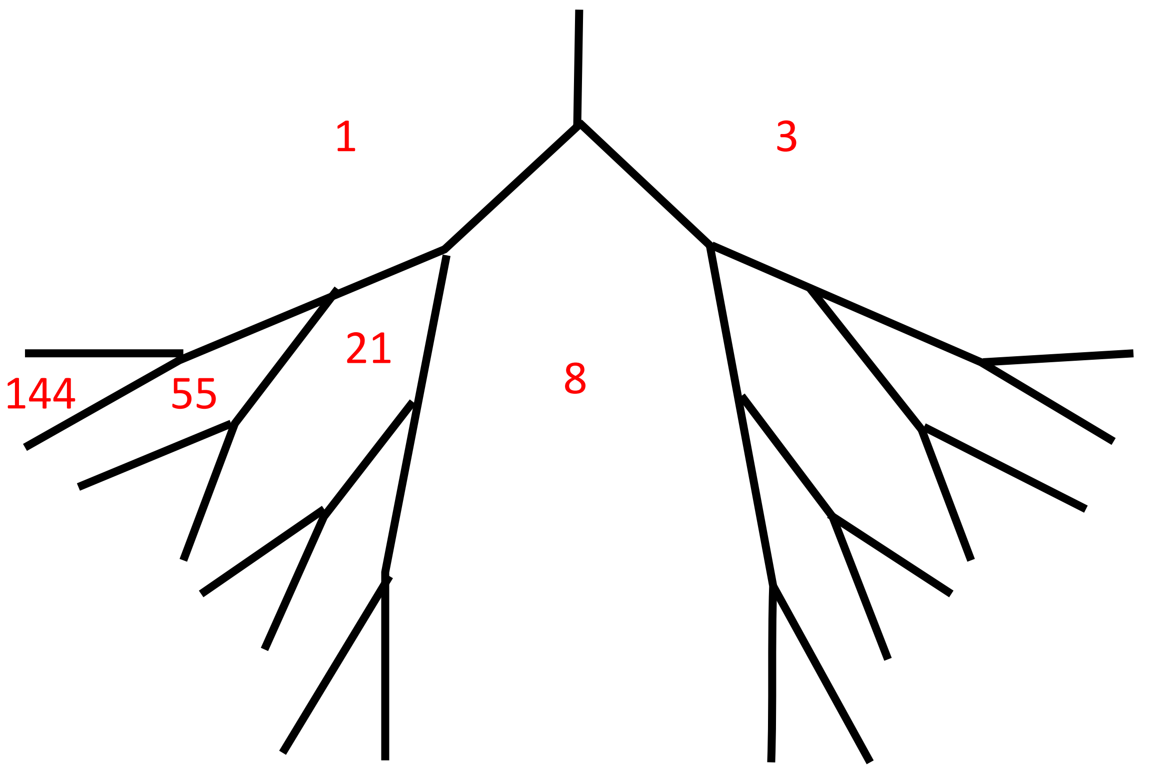
Markov tree ($x^2+y^2+z^2=3xyz+2$)



Markov tree (z’=3xy-z)



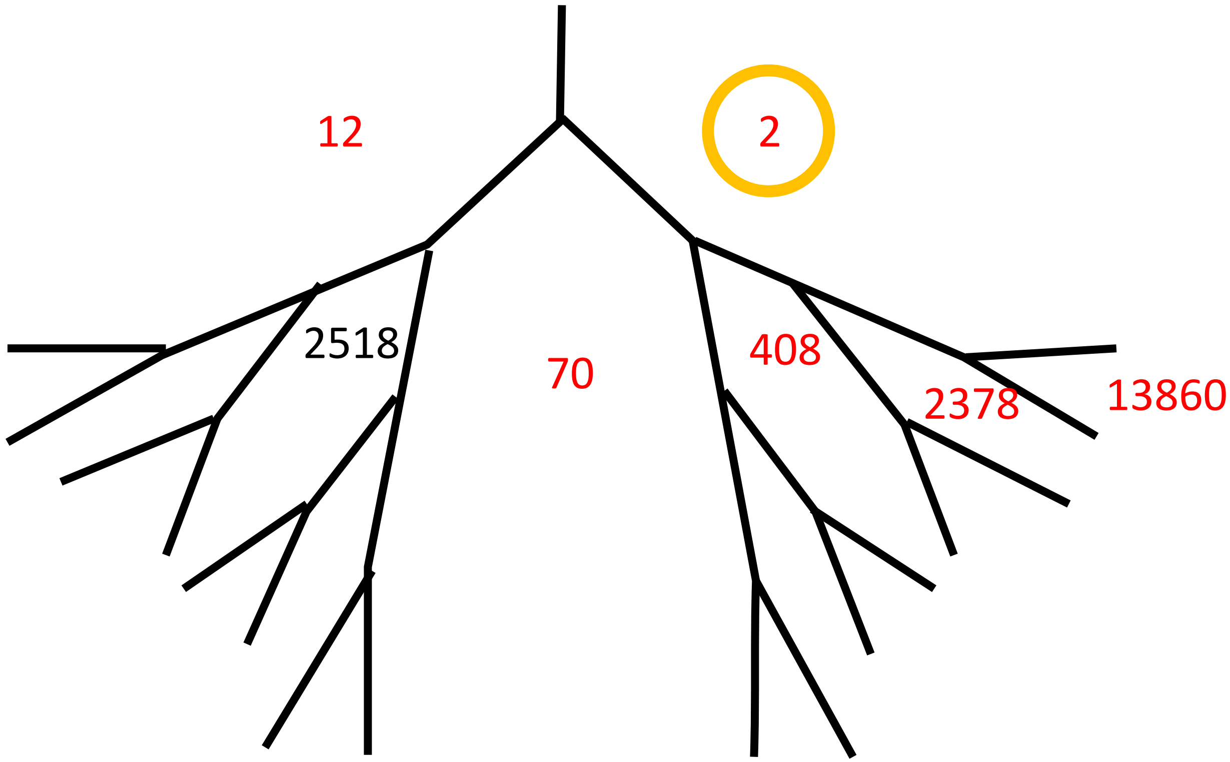
Markov tree (λ with all [1,2])



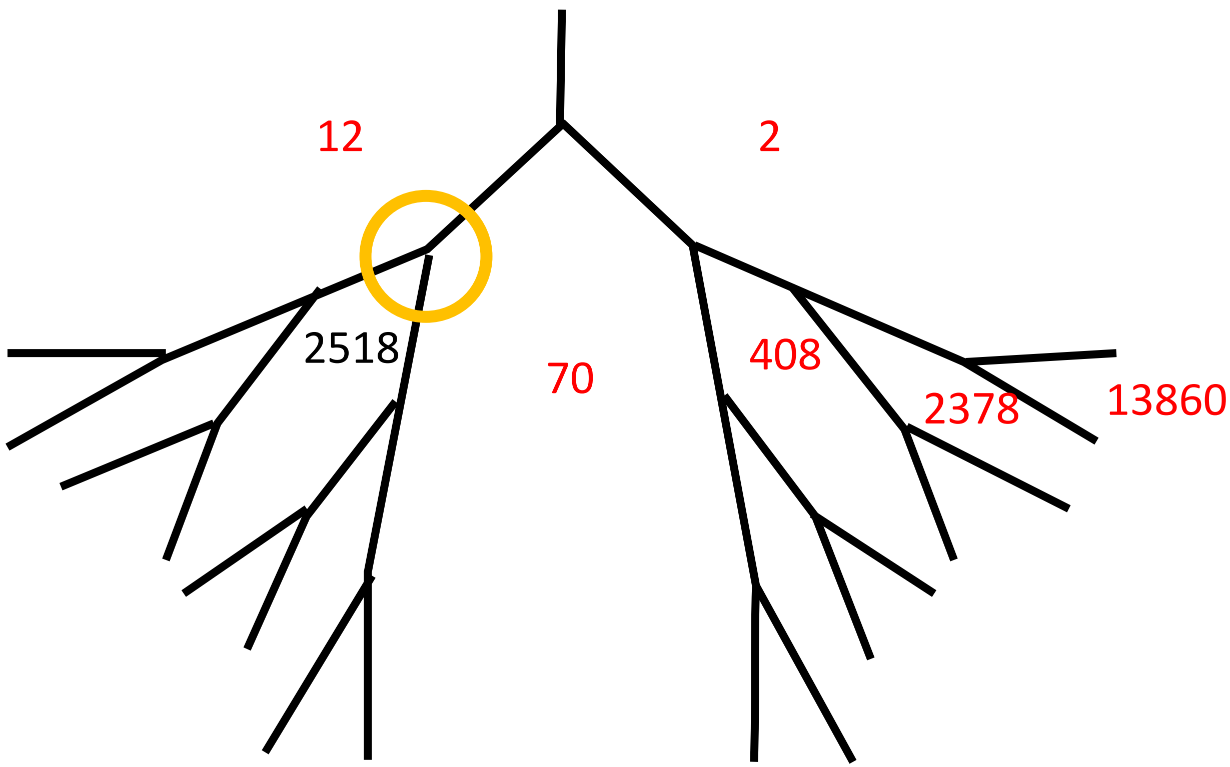
Pell numbers

Pell numbers, P[n+1]=2P[n]+P[n-1], P1=1, P2=2										
P[n]	**1**	**2**	**5**	**12**	**29**	**70**	**169**	**408**	**985**	**985**
P[2n-1]	1		5		29		169		985	
P[2n]		**2**		**12**		**70**		**408**		**2378**
P[n]	**5741**	**13860**	**33461**	**80782**	**195025**	**470832**	**1136689**	**2744210**	**6625109**	**15994428**
P[2n-1]	5741		33461		195025		1136689		6625109	
P[2n]		**13860**		**80782**		**470832**		**2744210**		**15994428**

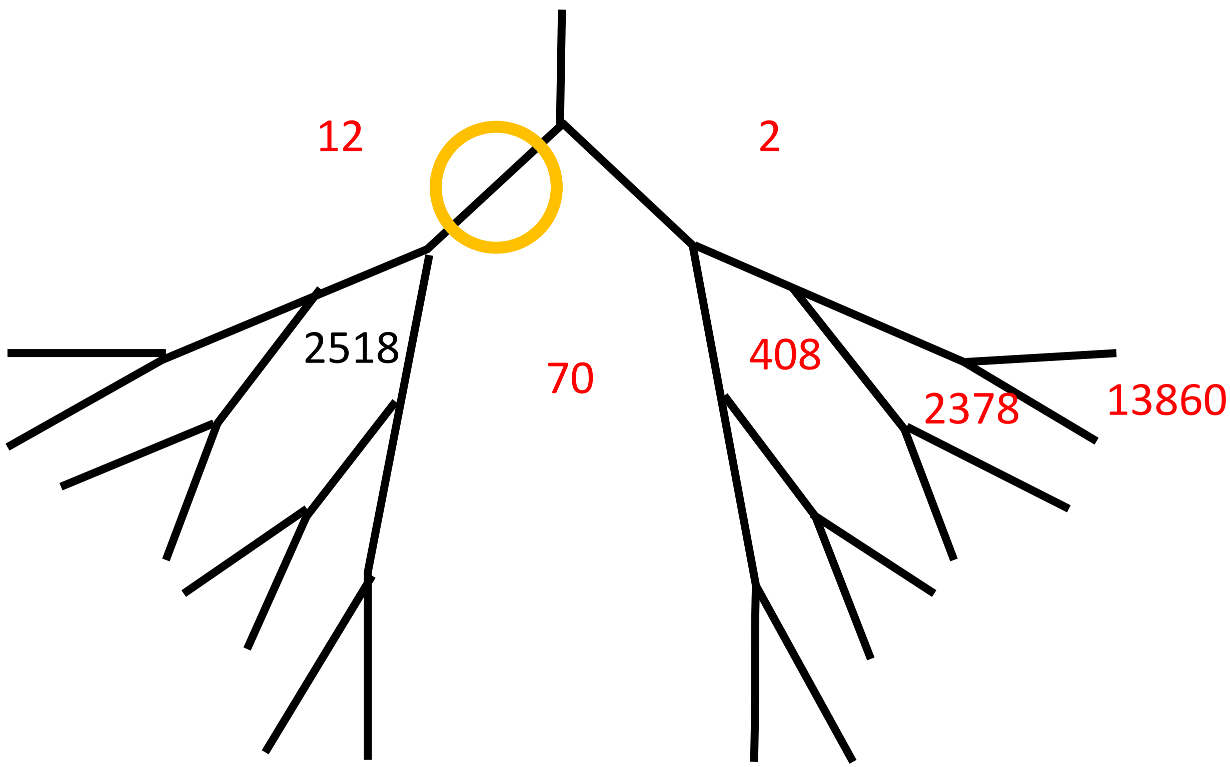
Markov tree (Pell)



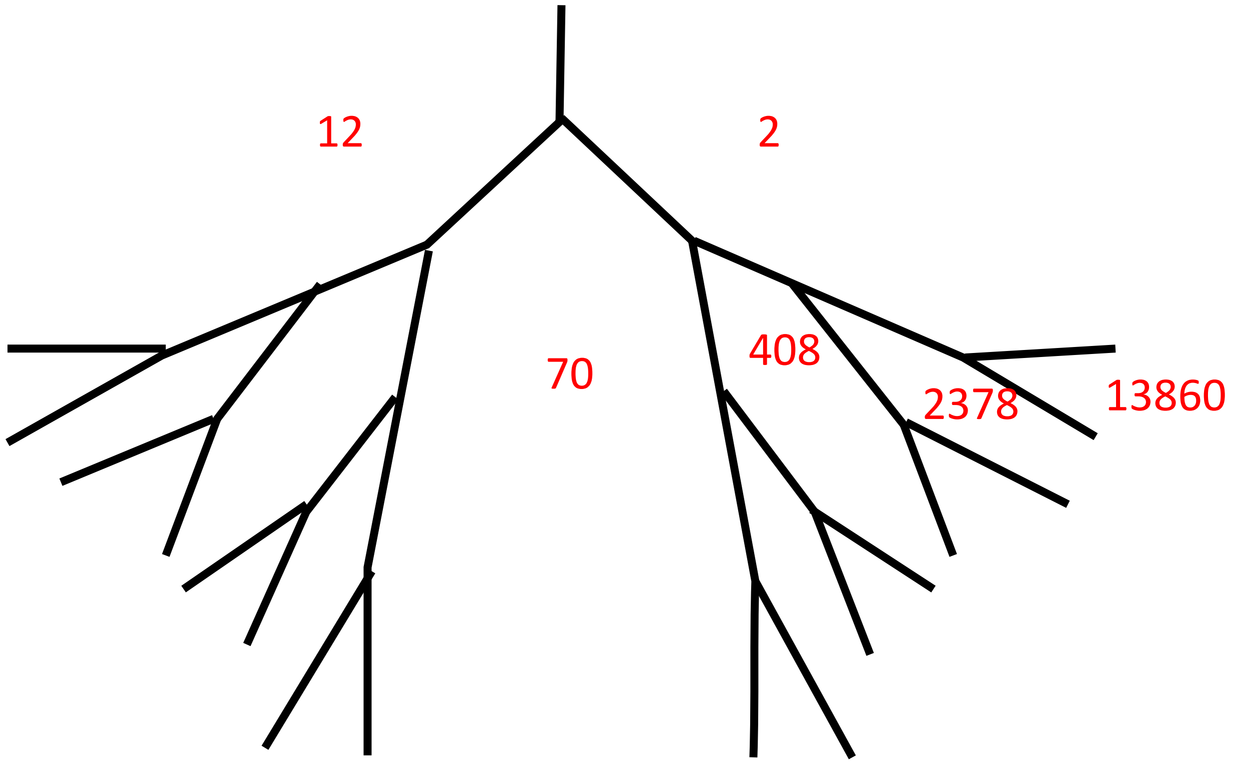
Markov tree ($x^2+y^2+z^2=3xyz+8$)



Markov tree (z’=3xy-z)



Markov tree (λ with all [1,2])



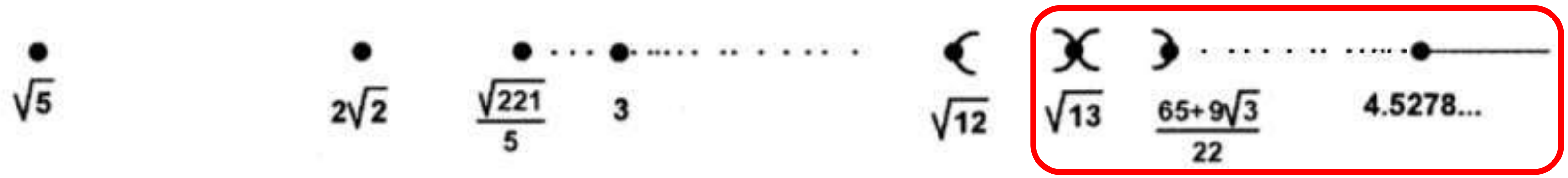
Results

- It is well known $\{F_{2n-1}\}$ and $\{P_{2n-1}\}$ appear in Markov Theorem ($\lambda=[\sqrt{5},3]$). {1,2,5,13,29,34,89,169,194,,,}
- $\{F_{2n}\}$ and $\{P_{2n}\}$ appear in $\lambda=[3,\sqrt{13}]$: mirror world of Markov theorem, {1,2,3,8,12,21,55,70,,,}
- On the viewpoints of $\{F_n\}$ and $\{P_n\}$, it is natural and within my expectation.
- Expanded Markov numbers does not match in anything table of OEIS database.
- I cannot tell it is already known or not yet.

Expanded Markov numbers

Markov numbers, x^2+y^2+z^2=3xyz+2/8										
M[n]	**1**	**2**	**3**	**8**	**12**	**21**	**55**	**70**	**144**	**377**
F[2n]	1		3	8		21	55		144	377
P[2n]		2			12			70		
M[n]	**408**	**987**	**2378**	**2584**	**5741**	**6765**	**13860**	**17711**	**46368**	**80782**
F[2n]		987		2584		6765		17711	46368	
P[2n]	408		2378		5741		13860			80782

Markov Lagrange spectrum



- α→λ=[3;2,1,2,1,,,]+[0;3,2,1,2,1,,,]
  →$(55+11\sqrt{3})/22+(10-2\sqrt{3})/22=(65+9\sqrt{3})/22$～3.6631
- 部分商がすべて3<1/2/3<すべて4<1/2/3/4<,,,

- Perron's gap: maximal open interval
  $(\sqrt{12}, \sqrt{13})$, $(\sqrt{13}, (65+9\sqrt{3})/22)$
- Hall's ray: everywhere dense, Freiman constant
  $\lambda>(2221564096+283748\sqrt{462})/491993569$～4.5278
- Almost decreasing gap size ($\lambda=\sqrt{13}\to(65+9\sqrt{3})/22\to\to\to$)

λ (Lagrange number) is a index of Diophantine approximation

- $\alpha=[a_0;a_1,a_2,a_3,,,]$, $|\alpha-p/q|<1/\lambda(q)^2$
- $\lambda=[a_{n+1};a_{n+2},a_{n+3},,,]+[0;a_n,a_{n-1},,,a_1]$

- λ, excluded from Markov numbers
- $\alpha\rightarrow\lambda=[3;2,1,2,1,,,]+[0;3,2,1,2,1,,,]$

  $\rightarrow(5+\sqrt{3})/2+2/(5+\sqrt{3})=(65+9\sqrt{3})/22\sim3.6631$
- $\alpha\rightarrow\lambda=[3;3,3,3,3,,,]+[0;2,1,2,1,2,1,,,]$

  $\rightarrow(3+\sqrt{13})/2+1/(1+\sqrt{3})=(2+\sqrt{13}+\sqrt{3})/2\sim3.6688$
- $\alpha\rightarrow\lambda=[4;3,1,3,1,,,]+[0;4,3,1,3,1,,,]$

  $\rightarrow(21+\sqrt{21})/6+6/(21+\sqrt{21})=(399+16\sqrt{21})/105\sim4.4982$

λ, excluded from Markov numbers

α	Continued fraction	f(p/q)	λ
ϕ	[1;1,1,1,1,1,,,]	(p+q)/p	$\sqrt{5}$
$\sqrt{2}$	[1;2,2,2,2,2,,,]	(p+2q)/(p+q)	$2\sqrt{2}$
$\sqrt{3}$	[1;1,2,1,2,1,2,,,]	(2p+3q)/(p+2q)	$2\sqrt{3}$
$\sqrt{5}$	[2;4,4,4,4,4,,,]	(2p+5q)/(p+2q)	$2\sqrt{5}$
$\sqrt{6}$	[2;2,4,2,4,2,4,,,]	(5p+12q)/(2p+5q)	$2\sqrt{6}$
$\sqrt{7}$	[2;1,1,1,4,1,1,1,4,,]	(8p+21q)/(3p+8q)	$2\sqrt{7}$
$\sqrt{8}$	[2;1,4,1,4,1,4,,,]	(3p+8q)/(p+3q)	$2\sqrt{8}$
$\sqrt{10}$	[3;6,6,6,6,6,,,]	(3p+10q)/(p+3q)	$2\sqrt{10}$

Total image of Markov tree, λ=[$\sqrt{5}$,$\sqrt{13}$]

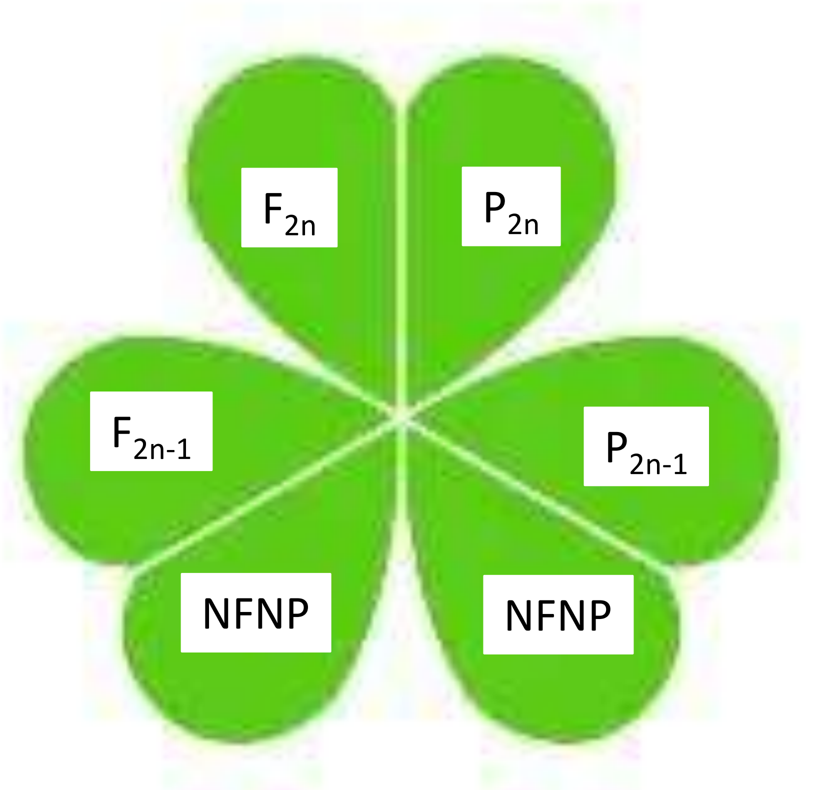
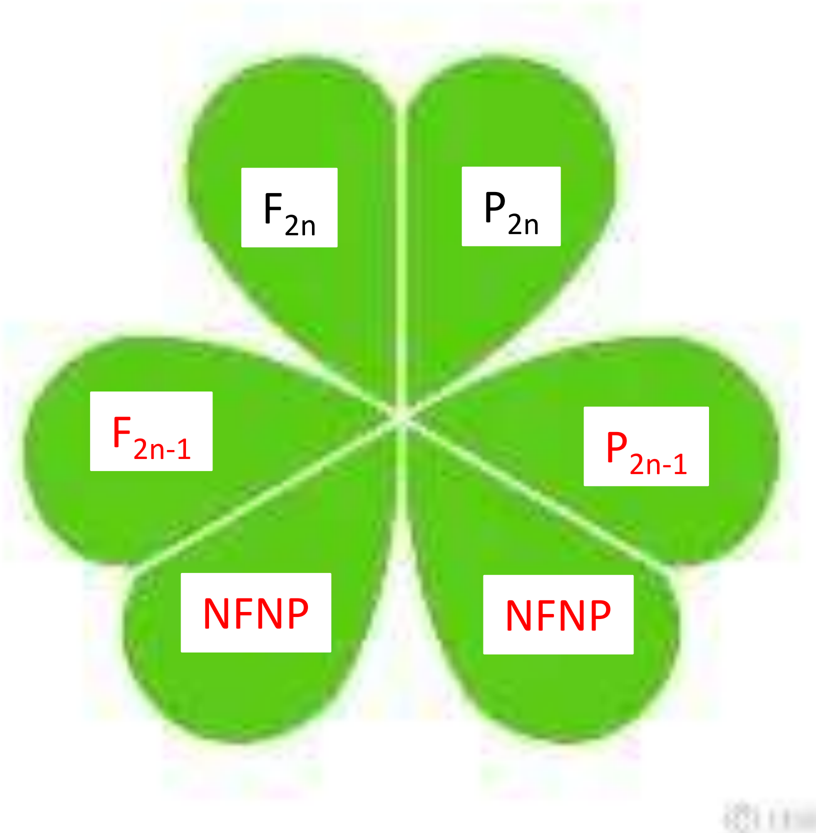


Image of Markov tree, $\lambda=[\sqrt{5},3]$



$\lambda=[\sqrt{5},3]$, $x^2+y^2+z^2=3xyz$, 完全二分木

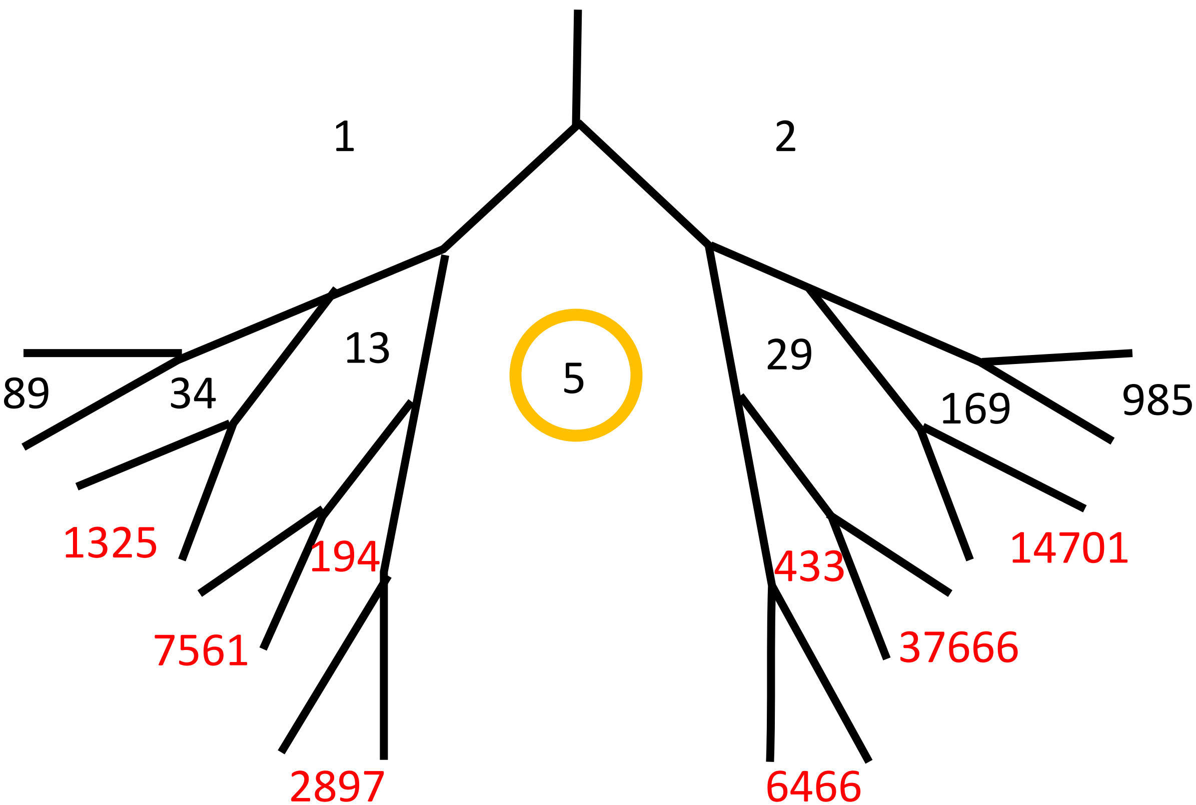
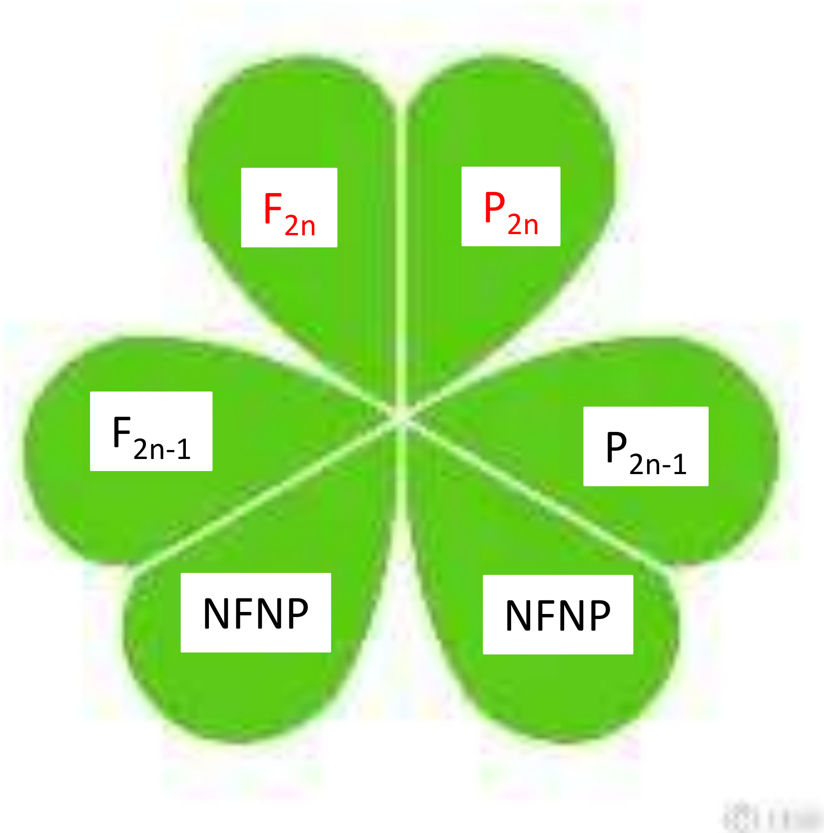
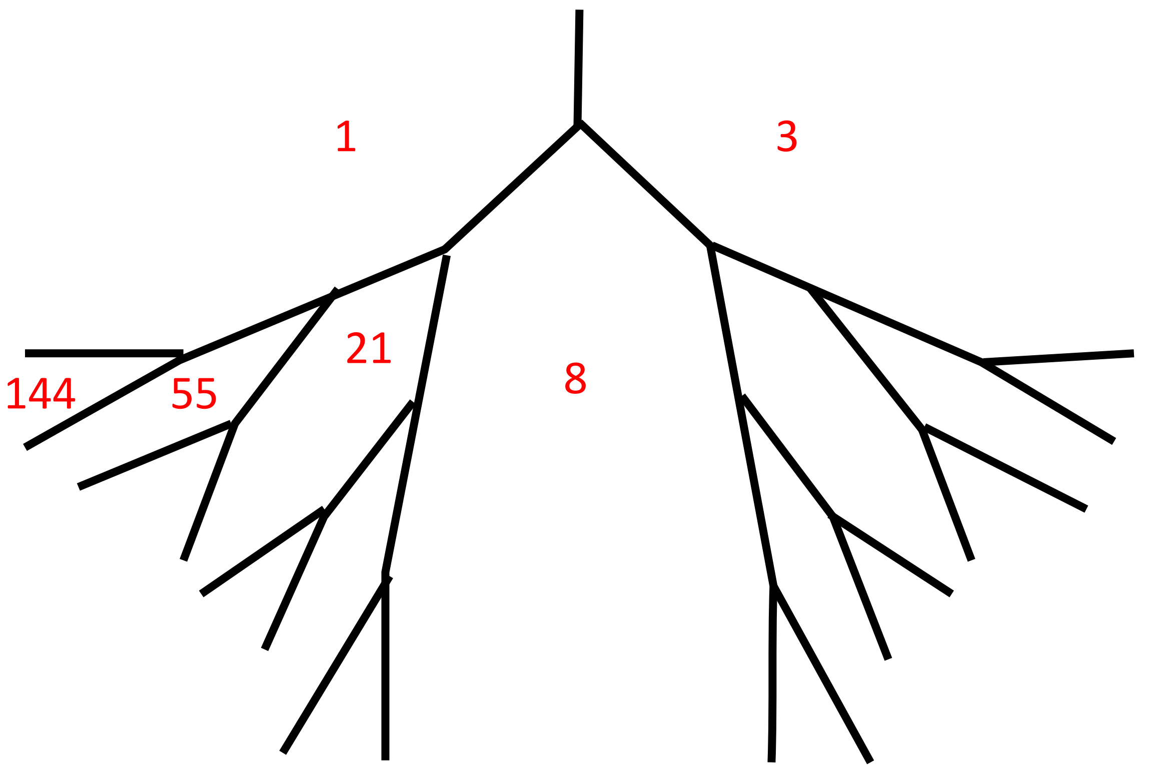


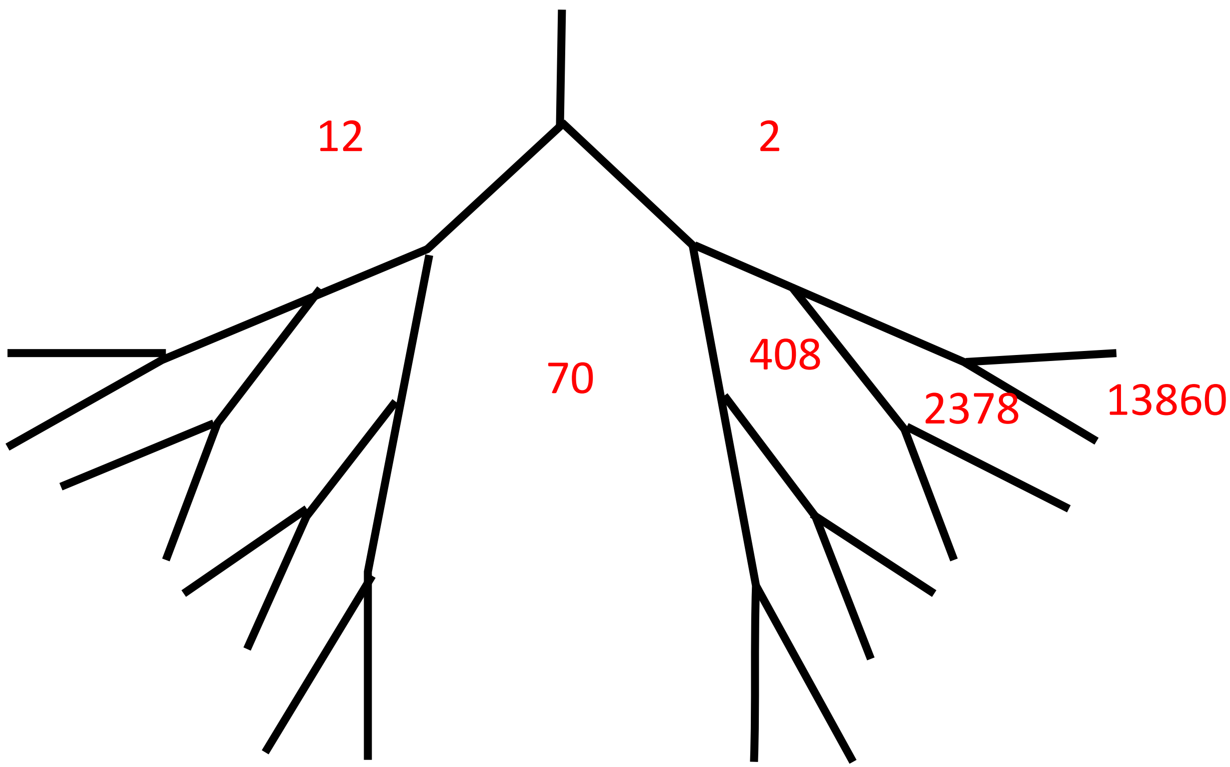
Image of Markov tree, $\lambda=[3,\sqrt{13}]$



$\lambda=[3,\sqrt{13}]$, $x^2+y^2+z^2=3xyz+2$, 幹→葉



$\lambda=[3,\sqrt{13}]$, $x^2+y^2+z^2=3xyz+8$, 幹→葉



Question asked by a high school student

- Please tell me the reason why Markov couldn't consider about $\lambda>3$ domain.

- How I should answer ?
- My answer for it; (incomplete, maybe, incorrect)
- $\lambda<3$: $x^2+y^2+z^2=3xyz$, $\lambda=(9x^2-4)^{1/2}/x$
- $\lambda>3$: $x^2+y^2+z^2=3xyz+2$
- $\lambda>3$: $x^2+y^2+z^2=3xyz+8$, $\lambda=(9x^2+4)^{1/2}/x$

Markov Theorem, $\lambda=[\sqrt{5},3]$

- Markov number: $\{F_{2n-1}\},\{P_{2n-1}\},\text{non}\{F_{2n-1}) \text{ non}\{P_{2n-1}\}$
- 1,2,5,13,29,34,89,169,194,233,433,610,985,,,

- Markov equation
- $x^2+y^2+z^2=3xyz$ (1879)

- Partial quotients of α are all [1,2]
- $\lambda=(9x^2-4)^{1/2}/x<3$ (1879)

Expanded Markov numbers, $\lambda=[3,\sqrt{13}]$

- Expanded Markov number (twin entity of Markov theorem) : $\{F_{2n}\}$, $\{P_{2n}\}$
- 1,2,3,8,12,21,55,70,144,377,408,987,,,

- Expanded Markov equation
- $x^2+y^2+z^2=3xyz+2$
- $x^2+y^2+z^2=3xyz+8$ (never known ?)

- Partial quotients of α are all [1,2]
- $\lambda=(9x^2+4)^{1/2}/x>3$ (1929)

Supplementary appendix

□を埋めよ

- 1,3,5,7,9,11,13,□,,,
- 1,2,4,8,16,32,64,□,,,
- 1,1,2,3,5,8,13,21,□,,,

- 1,2,2,3,3,3,4,4,4,4,5,5,5,5,5,,,
- 1,2,4,5,7,9,10,12,14,16,17,19,21,23,25,,,

□を埋めよ

- 1,3,5,7,9,11,13,□,,,
- 1,2,4,8,16,32,64,□,,,
- 1,1,2,3,5,8,13,21,□,,,

- 1,2,2,3,3,3,4,4,4,4,5,5,5,5,5,,,
- 1,2,4,5,7,9,10,12,14,16,17,19,21,23,25,,,
- $a_n=[\{1+(8n-7)^{1/2}\}/2]$
- $b_n=2n-[\{1+(8n-7)^{1/2}\}/2]$

□を埋めよ

- 1,3,5,7,9,11,13,□,,,
- 1,2,4,8,16,32,64,□,,,
- 1,1,2,3,5,8,13,21,□,,,

- 22,22,22,22,22,22,22,22,□,,,
- 1,11,21,1211,111221,312211,13112221,
  1113213211,31131211131221,□,,,

□を埋めよ

- 1,11,21,1211,111221,312211,13112221,
  1113213211,31131211131221,□,,,

- $\log c_{n+1}/\log c_n \sim (1.303577269\ldots)$
- $c_n \sim 10\hat{}(1.303577269\ldots)^n$
- $x^{71}-x^{69}-2x^{68}-x^{67}+2x^{66}+2x^{65}+x^{64}-x^{63}-x^{62}-x^{61}-x^{60}-x^{59}+\ldots+5x^9+x^7-7x^6+7x^5-4x^4+12x^3-6x^2+3x-6=0$